The Exponential Function

Suppose the newspaper headlines read, "The cost of living went up 10% this year." Can we translate this information into an *equation*? Let "V" denote the value of a dollar, in terms of the "real goods" it can buy — whatever economists mean by that. Let the elapsed time t be measured in years (y). Then suppose that V is a function of t, $V(t)$, which function we would like to know explicitly. Call *now* "$t = 0$" and let the initial value of the dollar (now) be V_0, which we could take to be \$1.00 if we disregard inflation at earlier times.[1]

Then our news item can be written

$$V(0) = V_0, \qquad \text{whereas} \qquad V(1\,\mathrm{y}) \;=\; (1 - 0.1)\,V_0 \;=\; 0.9\,V_0.$$

This formula can be rewritten in terms of the *changes* in the dependent and independent variables, $\Delta V = V(1\,\mathrm{y}) - V(0)$ and $\Delta t = 1\,\mathrm{y}$:

$$\frac{\Delta V}{\Delta t} = -0.1\,V_0, \tag{1}$$

where it is now to be *understood* that V is measured in "1998 dollars" and t is measured in years. That is, the average *time rate of change* of V is proportional to the value of V at the beginning of the time interval, and the constant of proportionality is $-0.1\ \mathrm{y}^{-1}$. (By y^{-1} or "inverse years" we mean the *per year* rate of change.)

This is almost like a derivative. If only Δt were infinitesimally small, it would *be* a derivative. Since we're just trying to describe the qualitative behaviour, let's make an *approximation*: assume that $\Delta t = 1$ year is "close enough" to an infinitesimal time interval, and that the above formula (1) for the inflation rate can be turned into an *instantaneous* rate of change:[2]

$$\frac{dV}{dt} = -0.1\,V. \tag{2}$$

This means that the dollar in your pocket right now will be worth only \$0.99999996829 in one second.

Well, this is interesting, but we cannot go any further with it until we ask a crucial question: "What will happen if this goes on?" That is, suppose we assume that equation (2) is not just a temporary situation, but *represents a consistent and ubiquitous property* of the function $V(t)$, the "real value" of your dollar bill as a function of time.[3]

Applying the d/dt "operator" to both sides of Eq. (2) gives

$$\frac{d}{dt}\left(\frac{dV}{dt}\right) = \frac{d}{dt}(-0.1\,V) \qquad \text{or} \qquad \frac{d^2V}{dt^2} = -0.1\,\frac{dV}{dt}. \tag{3}$$

[1] Since our dollar will be worth *less* a year from now, we should really call it **de**flation!

[2] The error introduced by this approximation is not very serious.

[3] Banks, insurance companies, trade unions, and governments all pretend that they don't assume this, but they would all go bankrupt if they *didn't* assume it.

But dV/dt is given by (2). If we substitute that formula into the above equation (3), we get

$$\frac{d^2V}{dt^2} = (-0.1)^2\, V = 0.01\, V. \tag{4}$$

That is, the rate of change of the rate of change is always positive, or the (negative) rate of change is getting *less* negative all the time.[4] In general, whenever we have a *positive second derivative* of a function (as is the case here), the *curve is concave upwards*. Similarly, if the second derivative were *negative*, the curve would be concave *downwards*.

So by noting the initial value of V, which is formally written V_0 but in this case equals \$1.00, and by applying our understanding of the "graphical meaning" of the first derivative (slope) and the second derivative (curvature), we can visualize the function $V(t)$ pretty well. It starts out with a maximum downward slope and then starts to level off as time increases. This general trend continues indefinitely. Note that while the function always decreases, it *never reaches zero*. This is because, the closer it gets to zero, the slower it decreases [see Eq. (2)]. This is a very "cute" feature that makes this function especially fun to imagine over long times.

We can also apply our analytical understanding to the formulas (2) and (4) for the derivatives: every time we take still another derivative, the result is still proportional to V — the constant of proportionality just picks up another factor of (-0.1). This is a *really neat* feature of this function, namely that we can write down *all its derivatives* with almost no effort:

$$\begin{aligned}
\frac{dV}{dt} &= -0.1\, V && (5)\\
\frac{d^2V}{dt^2} &= (-0.1)^2\, V = +0.01\, V && (6)\\
\frac{d^3V}{dt^3} &= (-0.1)^3\, V = -0.001\, V && (7)\\
\frac{d^4V}{dt^4} &= (-0.1)^4\, V = +0.0001\, V && (8)\\
&\vdots \\
\frac{d^nV}{dt^n} &= (-0.1)^n\, V \qquad \text{for } \textit{any } n. && (9)
\end{aligned}$$

This is a pretty nifty function. What *is* it? That is, can we write it down in terms of familiar things like t, t^2, t^3, and so on?

First, note that Eq. (9) can be written in the form

$$\frac{d^nV}{dt^n} = k^n\, V, \qquad \text{where} \qquad k = -0.1 \tag{10}$$

A simpler version would be where $k = 1$, giving

$$\frac{d^nW}{dt^n} = W, \tag{11}$$

[4] A politician trying to obfuscate the issue might say, "The rate of decrease is decreasing."

$W(t)$ being the function satisfying this criterion. We should perhaps try figuring out this simpler problem first, and then come back to $V(t)$.

Let's try expressing $W(t)$, then, as a linear combination[5] of such terms. For starters we will try a "third order polynomial" (*i.e.* we allow terms up to t^3):

$$\begin{aligned} W(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3. \qquad \text{Then} \\ \frac{dW}{dt} &= a_1 + 2a_2 t + 3a_3 t^2 \end{aligned}$$

follows by simple "differentiation" [a single word for "taking the derivative"]. Now, these two equations have similar-looking right-hand sides, provided that we pretend not to notice that term in t^3 in the first one, and provided the constants a_n obey the rule $a_{n-1} = n a_n$ [*i.e.* $a_0 = a_1$, $a_1 = 2a_2$ and $a_2 = 3a_3$]. But we can't really neglect that t^3 term! To be sure, its "coefficient" a_3 is smaller than any of the rest, so if we had to neglect anything it might be the best choice; but we're trying to be precise, right? How precise? Well, precise enough. In that case, would we be precise enough if we added a term $a_4 t^4$, preserving the rule about coefficients [$a_3 = 4a_4$]? No? Then how about $a_5 t^5$? And so on. No matter how precise an agreement with Eq. (11) we demand, we can always take enough terms, using this procedure, to achieve the desired precision. Even if you demand infinite precision, we just [just?] take an infinite number of terms:

$$W(t) = \sum_{n=0}^{\infty} a_n\, t^n, \qquad \text{where} \qquad a_{n-1} = n\, a_n \qquad \text{or} \qquad a_n = \frac{a_{n-1}}{n}. \tag{12}$$

Now, suppose we give $W(t)$ the initial value 1. [If we want a different initial value we can just multiply the whole series by that value, without affecting Eq. (11).] Well, $W(0) = 1$ tells us that $a_0 = 1$. In that case, $a_1 = 1$ also, and $a_2 = \frac{1}{2}$, and $a_3 = \frac{1}{2} \times \frac{1}{3}$, and $a_4 = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}$, and so on. If we define the *factorial* notation,

$$n! \equiv n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1 \tag{13}$$

(read, "*n factorial*") and define $0! \equiv 1$, we can express our function $W(t)$ very simply:

$$W(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tag{14}$$

We could also write a more abstract version of this function in terms of a generalized variable "x":

$$W(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{15}$$

Let's do this, and then define $x \equiv k\,t$ and set $V(t) = V_0\, W(x)$. Then, by the CHAIN RULE for derivatives,[6]

$$\frac{dV}{dt} = V_0 \frac{dW}{dx} \frac{dx}{dt} \tag{16}$$

[5] "Linear combination" means we multiply each term by a simple constant and add them up.

[6] The CHAIN RULE for derivatives says that if z is an explicit function of y, $z(y)$, and y is an explicit function of x, $y(x)$, then z is an *implicit* function of x and its derivative with respect to x is given by

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

and since $\frac{d}{dt}(k\,t) = k$, we have

$$\frac{dV}{dt} \;=\; k\,V_0\,W \;=\; k\,V. \tag{17}$$

By repeating this we obtain Eq. (10). Thus

$$V(t) \;=\; V_0\,W(kt) \;=\; V_0 \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} \tag{18}$$

where $k = -0.1$ in the present case.

This is a nice description; we can always calculate the value of this function to any desired degree of accuracy by including as many terms as we need until the change produced by adding the next term is too small to worry us.[7] But it is a little clumsy to keep writing down such an unwieldy formula every time you want to refer to this function, especially if it is going to be as popular as we claim. After all, mathematics is the art of precise abbreviation. So we give $W(x)$ [from Eq. (15)] a special name, the "**exponential**" function, which we write as either[8]

$$\exp(x) \qquad \text{or} \qquad e^x. \tag{19}$$

In FORTRAN it is represented as `EXP(X)`. It is equal to the number

$$e \;=\; 2.718281828459045 09\cdots \tag{20}$$

raised to the x^{th} power. In our case we have $x \equiv -0.1\,t$, so that our "answer" is

$$V(t) \;=\; V_0\,e^{-0.1\,t} \tag{21}$$

which is a lot easier to write down than Eq. (18).

Now, the choice of notation e^x is not arbitrary. There are a lot of rules we know how to use on a number raised to a power. One is that

$$e^{-x} \;\equiv\; \frac{1}{e^x} \tag{22}$$

You can easily determine that this rule also works for the definition in Eq. (15).

The "inverse" of this function (the power to which one must raise e to obtain a specified number) is called the "**natural logarithm**" or "ln" function. We write

$$\text{if} \qquad W = e^x, \qquad \text{then} \qquad x = \ln(W)$$

or

$$x \;=\; \ln(e^x) \tag{23}$$

A handy application of this definition is the rule

$$y^x = e^{x\ln(y)} \qquad \text{or} \qquad y^x = \exp[x\ln(y)]. \tag{24}$$

[7] This is exactly what a "scientific" hand calculator does when you push the function key whose name will be revealed momentarily.

[8] Now you know which key it is on a calculator.

Before we return to our original function, is there anything more interesting about the "natural logarithm" than that it is the inverse of the "exponential" function? And what is so all-fired special about e, the "base" of the natural log? Well, it can easily be shown[9] that the *derivative* of $\ln(x)$ is a very simple and familiar function:

$$\frac{d[\ln(x)]}{dx} = \frac{1}{x}. \tag{25}$$

This is perhaps the most useful feature of $\ln(x)$, because it gives us a direct connection between the exponential function and a function whose derivative is $1/x$. [The handy and versatile rule $\frac{d(x^r)}{dx} = rx^{r-1}$ is valid for any value of r, including $r = 0$, but it doesn't help us with this task. Why?] It also explains what is so special about the number e.

Summary: The Exponential Function(s)

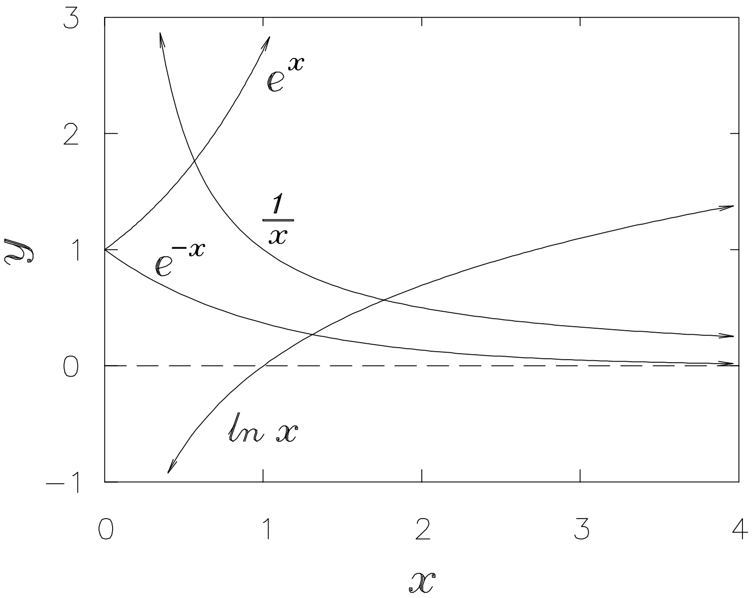

Figure 5.1 The functions e^x, e^{-x}, $\ln(x)$ and $1/x$ plotted on the same graph over the range from $x = 0$ to $x = 4$. Note that $\ln(0)$ is undefined. [There is no finite power to which we can raise e and get zero.] Similarly, $1/x$ is undefined at $x = 0$, while $1/(-x) = -1/x$. Also, $\ln(1) = 0$ [because any number raised to the zeroth power equals 1 — you can easily check this against the definitions] and $\ln(\xi)$ [where ξ any positive number less than 1] is negative. However, there is no such thing as the natural logarithm of any *negative* number.

Our formula (21) for the real value of your dollar as a function of time is the *only* function which will satisfy the differential equation (2) from which we started. The *exponential* function is one of the most useful of all for solving a wide variety of differential equations. For now, just remember this:

[9] Watch for this phrase! Whenever someone says "It can easily be shown. . . ," they mean, "This is possible to prove, but I haven't got time; besides, I might want to assign it as homework."

Whenever you have $\frac{dy}{dx} = k\,y$, you can be sure that $y(x) = y_0\,e^{kx}$ where y_0 is the "initial value" of y [when $x = 0$]. Note that k can be either positive or negative.

Finally, note the property of the *second* derivative:

$$\frac{d^2y}{dx^2} = k^2\,y. \tag{26}$$

We will see another equation almost like this when we talk about SIMPLE HARMONIC MOTION.

An Example from Mechanics: Damping

We should really work out at least one example applying the exponential function to a real Mechanics problem. The classic example is where an object (mass m) is moving with an initial velocity v_0, starting from an initial position x_0, and experiences a *frictional damping force* F_d which is *proportional to the velocity* and (as always, for frictional forces) in the direction opposite to the velocity: $F_d = -\kappa\,v$. The equation of motion then reads $a = -(\kappa/m)\,v$ or

$$\frac{d^2x}{dt^2} = -k\,\frac{dx}{dt} \tag{27}$$

where we have combined κ and m into the constant $k \equiv \kappa/m$. This can also be written in the form

$$\frac{dv}{dt} = -k\,v$$

which should ring a bell! The solution (for the velocity v) is

$$v(t) = v_0\,e^{-k\,t} \tag{28}$$

To obtain the solution for $x(t)$, we switch back to the notation

$$\frac{dx}{dt} = v_0\,e^{-k\,t} \qquad \Longrightarrow \qquad \int_{x_0}^{x} dx = v_0 \int_0^t e^{-k\,t}\,dt$$

and note that the function whose time derivative is $e^{-k\,t}$ is $-\frac{1}{k}\,e^{-k\,t}$, giving

$$x - x_0 = -\frac{v_0}{k}\left[e^{-k\,t}\right]_0^t$$

where the $[\cdots]_0^t$ notation means that the expression in the square brackets is to be "evaluated between 0 and t" — *i.e.* plug in the upper limit (just t itself) for t in the expression and then *subtract* the value of the expression with the lower limit (0) substituted for t. In this case the lower limit gives $e^{-0} = e^0 = 1$ (anything to the zeroth power gives one) so the result is

$$x(t) = x_0 + \frac{v_0}{k}\left(1 - e^{-k\,t}\right) \tag{29}$$

The qualitative behaviour is plotted in Fig. 5.2. Note that $x(t)$ approaches a fixed "asymptotic" value $x_{\max} = x_0 + v_0/k$ as $t \to \infty$. The generic function $(1 - e^{-k\,t})$ is another useful addition to your pattern-recognition repertoire.

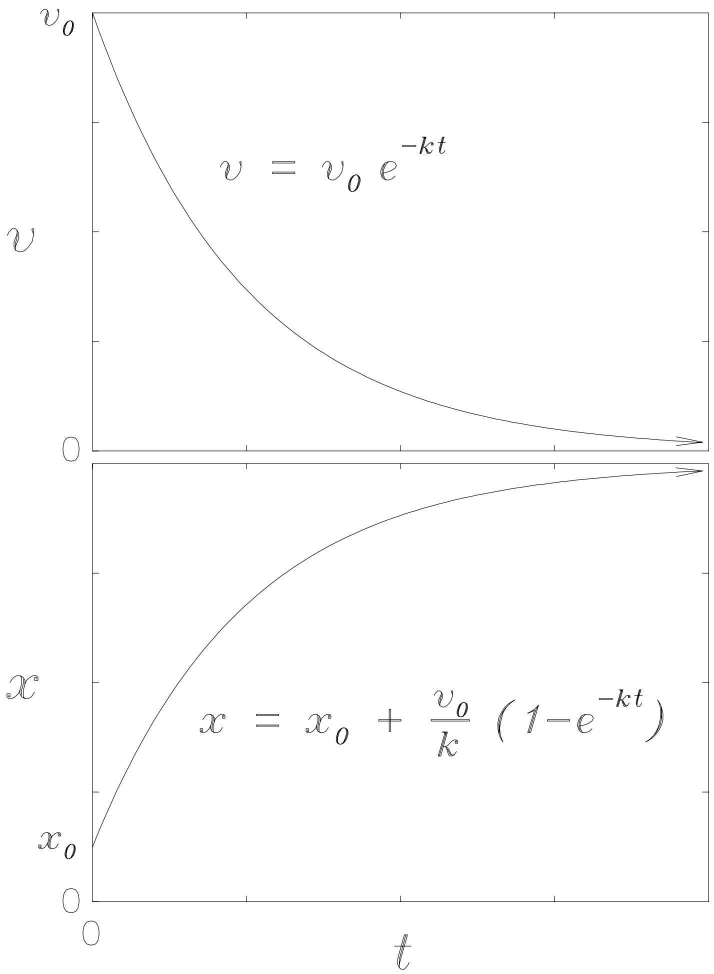

Figure 5.2 Solution to the *damping force* equation of motion, in which the frictional force is proportional to the velocity.