Force *vs.* Mass

"If I have seen further than other men, it is because I stood on the shoulders of giants." – Isaac Newton

Isaac Newton (1642-1727) published his masterwork, *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy") in 1687. In this tome he combined the individually remarkable conceptual achievements of calculus, vectors and an elegant expression of the simple relationship between *force* and *inertia* (which in effect gave definition to those entities for the first time) to produce an integrated description of the interactions between objects and exactly how they produce different kinds of motion. This was the true beginning of the science of *dynamics*, for it marked the adoption of the *descriptive paradigms* that are still used universally to describe dynamics, even after Quantum Mechanics has exposed Newtonian Mechanics as fundamentally inadequate.[1] Newton, like most great thinkers, had a variety of ludicrous foibles and was often a jerk in his dealings with others. I will not attempt to document his personal life, though many have done so [you can consult their work]; although it is interesting and revealing, it doesn't matter to our understanding of the conceptual edifice he built in the *Principia*. Moreover, I will make no attempt to introduce concepts in the order that Newton did, nor will I hesitate to use a more modern notation or even an updated version of a paradigm, with the rationale that (a) what matters most is getting the idea across clearly; and (b) we may have actually achieved a more elegant, compact understanding than Newton in the intervening centuries. This is one of the endearing (to me) traditions of Physics – and indeed of all genuine pursuit of truth[2] – we treasure an *æsthetic* of searching for a better, more elegant, more reliable, more accurate (with regard to predicting the results of experiments), *truer* model of the world and rooting out the demonstrably wrong parts of existing models. A frightening number of people who claim to *know* the Truth share no such æsthetic and in fact are dedicated to suppressing such activities when they threaten their most cherished and unexamined Truths. Grrrr....

[1]Note that Quantum Mechanics does *not* "prove Newtonian Mechanics wrong;" it merely reveals its shortcomings and the limits of its straightforward applicability. All paradigms have such shortcomings and limits, even Quantum Mechanics! Bridges did not fall down when Quantum Mechanics was "discovered," nor did engines or electromagnetic devices cease to function; we simply learned that Newtonian Mechanics and electromagnetic theory were *approximations* to a more fundamentally accurate picture furnished by Quantum Mechanics and Relativity, and where the approximation was no longer adequate to give a qualitatively correct description of the actual behaviour of matter.

Before we go on to expound Newton's "Laws" in their modern form it is useful to examine the "self-evident" [oh, yeah?] concepts of *force* and *mass* and their relationship with that relatively rigorously defined kinematic quantity, the *acceleration*.

9.1 Inertia *vs.* Weight

Prior to Newton, people who thought about such things observed that objects which had lots of *inertia* [i.e. were hard to get moving by pushing on them, even where a nearly frictionless horizontal motion was possible] were also invariably *heavy* [i.e. were pulled down toward the centre of the Earth with great force]. It was therefore understandable for them to have *equated* inertia with *weight*, the magnitude of the force of attraction to the Earth.[3] Newton was among

[2]I should say "truth" or otherwise indicate that I don't mean there *is* some sort of ultimate Truth that we can discover and then relax.

[3]It is of course easy for us to see the error of such thinking, because we are privy to Newton's paradigms; this should not

the first to suggest that *inertia* and *weight* were not necessarily the same thing, but that in fact the Earth's gravity *just happened* to pull down on objects with a force proportional to their inertial factor or "mass" (m) which was actually defined in terms of their resistance to horizontal acceleration by some force other than gravity.

9.1.1 The Eötvös Experiment

Is there any way to *test* Newton's conjecture that "inertial mass" (the quantitative measure of an objects resistance to acceleration by an applied force) is different from "gravitational mass" (the factor determining the weight of said object)? Certainly. But first we must make the proposition more explicit:

- Inertial mass m_I is an *additive property* of matter. That is, two identical objects, when combined, will have twice the inertial mass of either one by itself.[4]
- When subjected to a given force $\vec{\mathbf{F}}$ [a vector quantity, since it certainly has both magnitude and direction], an object will be accelerated in the direction of $\vec{\mathbf{F}}$ at a rate $\vec{\mathbf{a}}$ which is *inversely proportional*[5] *to its inertial mass* m_I. Mathematically,
$$\vec{\mathbf{a}} \propto \frac{\vec{\mathbf{F}}}{m_I}. \tag{1}$$
- Gravitational mass m_G is also an additive property of matter.
- The force of gravity $\vec{\mathbf{W}}$ pulling an object "down" toward the centre of the Earth (i.e. its *weight*) is proportional to its gravitational mass m_G. Let's write the constant of proportionality "g" so that $W = g\, m_G$ (where $W \equiv |\vec{\mathbf{W}}|$ is the *magnitude* of the weight, which is usually all we need, knowing as we do which way is "down") – or, in full vector notation,
$$\vec{\mathbf{W}} = -g\, m_G\, \hat{\mathbf{r}} \tag{2}$$
(where $\hat{\mathbf{r}}$ is the unit vector pointing *from* the centre of the Earth *to* the object in question).

The *combination* of the last two postulates is easy to check using a simple *balance*. However, it is not so easy to *separately* check these two propositions. See why? Fortunately, we don't have to.

If we put together the two equations $\vec{\mathbf{a}} \propto \vec{\mathbf{F}}/m_I$ and $\vec{\mathbf{W}} = -g\, m_G\, \hat{\mathbf{r}}$, noting that, in the case of the force of gravity *itself*, $\vec{\mathbf{F}} \equiv \vec{\mathbf{W}}$, we get
$$\vec{\mathbf{a}} \propto -\hat{\mathbf{r}}\, g\, \frac{m_G}{m_I} \tag{3}$$
– i.e. the acceleration *due to gravity* is in the $-\hat{\mathbf{r}}$ direction (towards the centre of the Earth), and is proportional to the *ratio* of the gravitational mass to the inertial mass. *So...* if the gravitational mass is *proportional to* the inertial mass, then *all objects should experience the same acceleration when falling due to the force of gravity*, at least in the absence of any other forces like air friction. Wait! Isn't this just what Galileo was always trying to tell us? Yep. *But was he right?*

Clearly the answer hangs on the proportionality of m_G and m_I. As we shall see, any nontrivial constant of proportionality can be absorbed into the definition of the *units* of force; thus instead of $\vec{\mathbf{a}} \propto \vec{\mathbf{F}}/m_I$ we can write $\vec{\mathbf{a}} = \vec{\mathbf{F}}/m_I$ and the question becomes, "Are inertial mass and gravitational mass *the same thing*?" The experimental test is of course to actually *drop* a variety

delude us into scorning the efforts of the "giants" on whose shoulders Newton stood to "see further than other men."

[4] This may seem absurdly self-evident, but in fact there *are* physical properties that are *not* additive! So we want to explicitly point out this assumption as a point of vulnerability of the model, in case it is found to break down later on. This sort of "full disclosure" is characteristic of any enterprise designed to get at the truth rather than to win an argument.

[5] This can be checked by applying a force to two identical objects stuck together and seeing if they accelerate exactly half as fast as either one individually subjected to the same force.

of objects in an evacuated chamber where there truly is no air friction (nor, we hope, any other more subtle types of friction) and measure their accelerations *as accurately as possible*. This was done by Eötvös to an advertised accuracy of 10^{-9} (one part per billion – often written 1 *ppb*) who found satisfactory agreement with Galileo's "law."[6] Henceforth I will therefore drop the $_G$ and $_I$ subscripts on mass and assume there is only one kind, *mass*, which I will write m.

9.1.2 Momentum

René Descartes and Christian Huygens together introduced the concept of *momentum* as the combination of an object's *weight* with its *velocity*, developing a rather powerful scheme for "before and after" analysis of isolated collisions and similar messy processes. I will be unfaithful to the historical sequence of conceptual evolution in this case primarily because I want to introduce the "impulse and momentum conservation law" later on as an example of the "emergence" of new paradigms from a desire to invent shortcuts around tedious mathematical calculations. Nevertheless, Newton actually formulated his Second Law in terms of momentum, so it would be too much of a distortion to omit at least a definition of momentum at this point, to wit:

$$\vec{\mathbf{p}} \equiv m\,\vec{\mathbf{v}} \tag{4}$$

I.e., the *momentum* of an object, a vector quantity which is almost always written $\vec{\mathbf{p}}$ (magnitude $|\vec{\mathbf{p}}| \equiv p$), is the *product* of the object's *mass* m and its vector *velocity* $\vec{\mathbf{v}}$.

[6] Recent re-measurements by Dicke *et al.* challenged Eötvös' ability to measure so accurately; they tentatively reported *deviations* from the expected results, suggesting that there might be an incredibly weak "fifth force" between the Earth and other matter that is different for protons than for neutrons. This was hot news for a while, but the excitement seems to have died down now, presumably due to new measurements that once again agree with Galileo and Eötvös.

9.2 Newton's Laws

We are now ready to state Newton's three "Laws" of motion, in Newton's own words:

1. FIRST LAW: *Every body continues in its state of rest, or of uniform motion in a right* [straight] *line, unless it is compelled to change that state by a force impressed on it.*
2. SECOND LAW: *The change in motion* [rate of change of momentum with time] *is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*
3. THIRD LAW: *To every action there is always opposed an equal reaction; or, the mutual actions of two bodies are always equal, and directed to contrary parts.*

Now, Newton's language was fairly precise, but to our modern ears it sounds a bit stilted and not very concise. We also imagine that, with the benefit of several centuries of practice, we have achieved a clearer understanding of these Laws than Newton himself. Regardless of the validity of this conceit, we like to express the Laws in a more modern form with a little mathematical notation thrown in:

1. FIRST LAW: *A body's velocity* $\vec{\mathbf{v}}$ [which might be zero] *will never change unless and until a force* $\vec{\mathbf{F}}$ *acts on the body.*
2. SECOND LAW: *The time rate of chage of the momentum of a body is* equal *to the force acting*

on the body. That is,

$$\frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}}. \tag{5}$$

3. THIRD LAW: *Whenever a force $\vec{\mathbf{F}}_{BA}$ is applied to A by B, there is an equal and opposite reaction force $\vec{\mathbf{F}}_{AB}$ on B due to A.* That is,

$$\vec{\mathbf{F}}_{AB} = -\vec{\mathbf{F}}_{BA}, \tag{6}$$

where the subscript $_{AB}$ (for instance) indicates the force *from A to B*.

As long as the mass m is constant[7] we have

$$\frac{d\vec{\mathbf{p}}}{dt} = \frac{d}{dt}(m\vec{\mathbf{v}}) = m\frac{d\vec{\mathbf{v}}}{dt} = m\,\vec{\mathbf{a}}$$

since the derivative of a constant times a variable is the constant times the derivative of the variable. Then the SECOND LAW takes the more familiar form,

$$\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}. \tag{7}$$

This famous equation is often written in scalar form,

$$\dot{p} \equiv \frac{dp}{dt} = F \qquad \text{or} \qquad F = m\,a$$

because $\dot{\vec{\mathbf{p}}}$ and $\vec{\mathbf{F}}$ are always in the same direction.

9.3 *What* Force?

The THIRD LAW is a real ringer. It looks so trivial, yet it warns us of a leading cause of confusion in mechanics problems: *There are always two forces* for every interaction! When A exerts a force $\vec{\mathbf{F}}_{AB}$ on B there is always an equal and opposite force $\vec{\mathbf{F}}_{BA} = -\vec{\mathbf{F}}_{AB}$ exerted back on A *by* B. The latter is arbitrarily designated the "reaction force," but of course this is only because we first started talking about the former; both forces have equal intrinsic status. So if you say, "The force between A and B is…." I don't know which force you are talking about! Never talk about "the force" unless you mean "the Force" from *Star Wars*. Always make up a *sentence* describing the *action* taking place: "The force *exerted on* [A] *by* [B] is…."

[7] Counterexamples are not as rare as you might think! Consider for instance a *rocket*, which is constantly losing mass as the motor burns fuel. In such cases the original form of the SECOND LAW is essential.

9.3.1 The Free Body Diagram

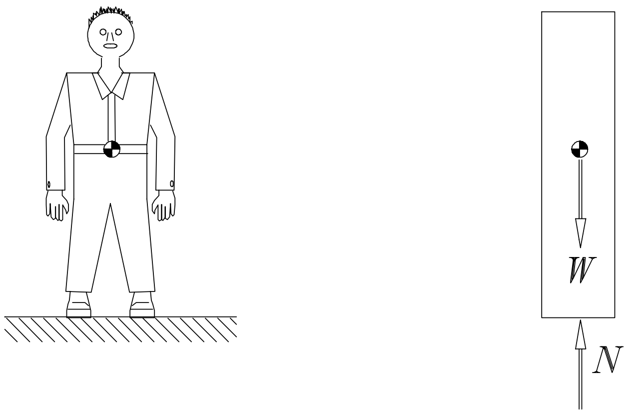

Figure 9.1 A man standing on the Earth (left) and his FBD (right). The man is pulled downward by the force of gravity W which is spread out over all his individual atoms but can be treated as if it were concentrated at his *centre of gravity* [CG] indicated on the diagram at about belt-buckle position. He is prevented from accelerating [falling] toward the centre of the Earth by the "normal force" N exerted upwards by the ground against his feet. These are the only two forces we need to consider to treat the problem of his *equilibrium* – i.e. the fact that he is not accelerating. The FBD on the right is perhaps a rather extreme example of a "simplified sketch" but it does serve the purpose, which is to show just the object in question and the forces acting on it *from outside*.

A good way to keep track of this (and cathect the right hemisphere in the process) is to draw what is universally known in Physics as a *Free Body Diagram* [FBD]. When you need to analyze the forces acting on a body [there are usually more than one!] the first step is to decide upon the *boundary* of "the body" – i.e. an imaginary surface that separates "the body" from "the outside world" so that we can talk unambiguously about who is applying which force to whom. Having done this in our imagination, it is usually wise to actually *draw* a little sketch of "the body" isolated from the rest of the world; it needn't be a good sketch, just a blob of approximately the right shape so we know what we are talking about. Then we draw in each of the vector *forces* acting *on* the body *from* other entities in the outside world; forces are always pictured as little arrows pointing in the direction of application of the force.[8] A rather trivial example is shown in Fig. 9.1. We call N a "normal" force because it is *normal* (perpendicular) to the horizontal surface on which he stands; this terminology (and the N symbol) will be extended to describe any force exerted by a *frictionless surface* [yes, I know, another idealization…], which can *only* be perpendicular to that surface. Think about it.

Atwood's Machine:

To illustrate the use of the FBD in nontrivial mechanics problems we can imagine another series of measurements[9] with a simple device known as Atwood's Machine. The apparatus is pictured in Fig. 9.2.

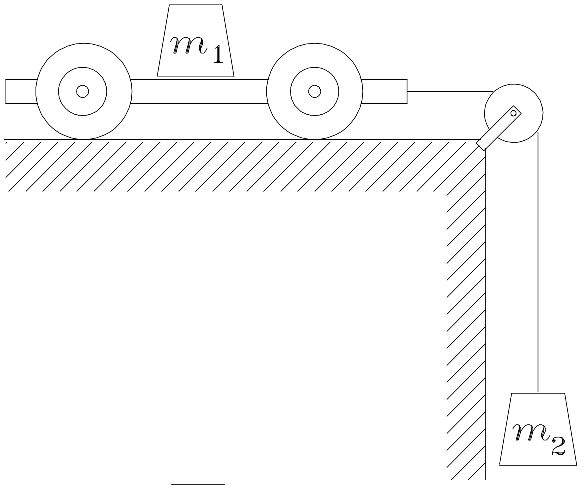

Figure 9.2 Atwood's Machine – one object labelled m_1 is glued to a massless cart with massless wheels that roll without friction on a perfectly horizontal surface. The cart is attached to a massless, unstretchable string that runs over yet another massless, frictionless pulley and is attached at the other end to a second object labelled m_2 that is pulled downward by the force of gravity. [You can see that a real experiment might involve a few corrections!] At the right are pictured the two separate FBD's for m_1 and m_2, showing all the external forces acting on each. Here W_1 is the *weight* of m_1 and N is the *normal force* exerted on m_1 by the horizontal surface (through the cart) to keep it from falling. Since it does *not* fall, N must exactly balance W_1. The only *unbalanced* force on m_1 is the *tension* T in the string, which accelerates it to the right. The tension in a string is the same everywhere, so the same T pulls *up* on m_2, partly counteracting its *weight* W_2.

It is easy to see that the two vertical forces (W_1 and N) acting on m_1 must cancel. The rest is less trivial. The weight of m_2 is given by $W_2 = m_2\, g$; thus for m_1 and m_2, respectively, we have

[8] If we mess up and draw the force in the opposite direction from its actual direction of application, we needn't worry, as the mathematics will automatically deliver up a result with a $-$ sign as if to say, "This force is in the opposite direction from the way you drew it, dummy!"

[9] Aha! another *Gedankenexperiment*! But this time we can actually imagine performing it in our basement – or in a teaching lab at the University (where in fact it is almost always one of the required experiments in every first year Physics course). Of course, the actual experiment is beset by numerous annoying imperfections that interfere with our cherished idealizations and require tedious and ingenious corrections. Even simple experiments are hard in real life!

the "equations of motion"

$$a_1 = \frac{T}{m_1} \qquad \text{(to the right)}$$

and

$$a_2 = \frac{m_2\, g - T}{m_2} \qquad \text{(downward).}$$

But we have here three unknowns (a_1, a_2 and T) and only two equations. The rules of linear algebra say that we need at least as many equations as unknowns to find a solution! Our salvation lies in recognition of the *constraints* of the system: Because the string does not stretch or go limp, both masses are *constrained* to move exactly the same distance (though in different directions) and therefore both experience the same *magnitude* of acceleration a. Thus our third equation is $a_1 = a_2 = a$ and we can equate the right sides of the two previous equations to get

$$\frac{T}{m_1} = \frac{m_2\, g - T}{m_2}$$

which we multiply through by $m_1\, m_2$ to get

$$m_2\, T = m_1\, m_2\, g - m_1\, T$$

$$\text{or} \qquad T\,[m_1 + m_2] = m_1\, m_2\, g$$

$$\text{or} \qquad T = \frac{m_1\, m_2\, g}{m_1 + m_2}.$$

Plugging this back into our first equation gives

$$a = g\, \frac{m_1}{m_1 + m_2}.$$

A quicker, simpler, more intuitive (and thus riskier) way of seeing this is to picture the pair of constrained masses *as a unit*. Let's use this approach to replace the distinction bewteen gravitational and inertial mass, just to see how it looks. The accelerating force is provided by the *weight* W_2 of m_2 which is given by $W_2 = g\, m_{2_G}$, where m_{2_G} is the gravitational mass of m_2. However, this force must accelerate *both* objects at the same rate because the string *constrains* both to move together (though in different directions). Thus the net inertia to be overcome by W_2 is the *sum* of the inertial masses of m_1 and m_2, so the acceleration is given by

$$a = \frac{W_2}{m_{1_I} + m_{2_I}} = g\, \frac{m_{2_G}}{m_{1_I} + m_{2_I}}$$

$$\text{or} \qquad \frac{a}{g} = \frac{m_{2_G}}{m_{1_I} + m_{2_I}}.$$

The latter form expresses the acceleration explicity in units of g, the acceleration of gravity, which is often called "one gee."

Suppose we have *three identical objects*, each of which has the *same* inertial mass m_I and the *same* gravitational mass m_G. [This can easily be checked using a balance and a standard force like a spring.] Then we use two of them for m_1 and m_2, set the apparatus in motion and measure the acceleration in "gees." The result will be $a/g = m_G/2m_I$. Next we put *two* of the objects on the *cart* and leave the third hanging. This time we should get $a/g = m_G/3m_I$. Finally we hand two and leave one on the cart, for $a/g = 2m_G/3m_I$. If the measured accelerations are actually in the ratios of $\frac{1}{2} : \frac{1}{3} : \frac{2}{3}$ then it must be true that m_G/m_I is constant – i.e. that m_G is proportional to m_I or that in fact they are really basically the same thing (in this case)! Unfortunately we have only confirmed this *for these three identical objects*. In fact all we have really demonstrated is that our original postulates are not trivially wrong. To go further we need to repeat the Eötvös experiment.