Celestial Mechanics

One of the triumphs of Newton's Mechanics was that he was able, using only his LAWS OF MOTION and a postulated UNIVERSAL LAW OF GRAVITATION, to explain the empirical LAWS OF PLANETARY MOTION discovered by Johannes Kepler. [Clearly there was a great deal more respect for Law in those days than there is now!] Although the phenomenology of *orbits* (circular, elliptical and hyperbolic) would appear to be rather esoteric and applicable *only* to astronomy [and, today, astrogation], in fact the paradigm of *uniform circular motion* (*i.e.* motion in a circle at constant speed) is one of the most versatile in Physics. Let us begin, therefore, by deriving its essential and characteristic features.

Circular Motion

Although no real orbit is ever a perfect circle, those of the inner planets aren't too far off and in any case it is a convenient idealization. Besides, we aren't restricted to planetary orbits here; the following derivation applies to *any* form of uniform circular motion, from tether balls on ropes to motorcycles on a circular track to charged particles in a cyclotron.[1]

Radians

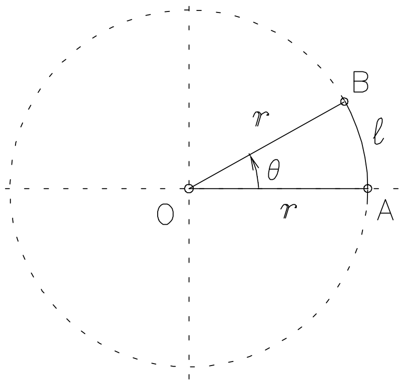

Figure 10.1 Definition of the angle $\theta \equiv \ell/r$.

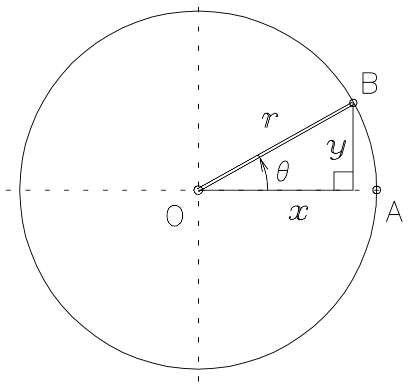

Figure 10.2 Definition of the trigonometric function $\cos(\theta) \equiv x/r$, $\sin(\theta) \equiv y/r$, $\sin(\theta) \equiv x/y$ *etc.*

Rate of Change of a Vector

The derivative of a *vector* quantity $\vec{\mathbf{A}}$ with respect to some independent variable x (of which it is a function) is defined in exactly the same way as the derivative of a *scalar* function:

$$\frac{d\vec{\mathbf{A}}}{dx} \equiv \lim_{\Delta x \to 0} \frac{\vec{\mathbf{A}}(x + \Delta x) - \vec{\mathbf{A}}(x)}{\Delta x} \qquad (1)$$

There is, however, a dramatic difference between scalar derivatives and vector derivatives: the latter can be nonzero *even if the* **magnitude** A of the vector $\vec{\mathbf{A}}$ *remains constant.* This is a consequence of the fact that vectors have *two* properties: magnitude and direction. If the *direction* changes, the derivative is nonzero, even if the *magnitude* stays the same!

This is easily seen using a sketch in two dimensions:

[1] You could even imagine examples from "outside Physics," in which the *radius* and *speed* were purely metaphorical; but I can't think of one. . . .

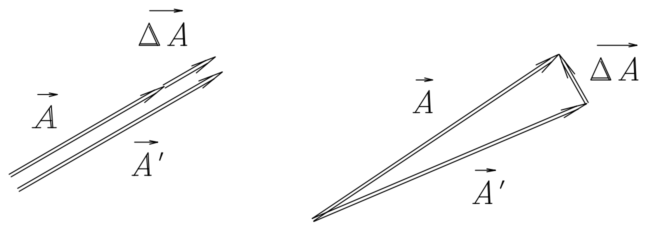

Figure 10.3

In the case on the left, the vector $\vec{\mathbf{A}}'$ is in the same direction as $\vec{\mathbf{A}}$ but has a different *length*. [The two vectors are drawn side by side for visual clarity; try to imagine that they are on top of one another.] The difference vector $\vec{\mathbf{\Delta A}} \equiv \vec{\mathbf{A}}' - \vec{\mathbf{A}}$ is *parallel* to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{A}}'$.[2] If we divide $\vec{\mathbf{\Delta A}}$ by the change Δx in the independent variable (of which $\vec{\mathbf{A}}$ is a function) and let $\Delta x \to 0$ then we find that the *derivative* $\frac{d\vec{\mathbf{A}}}{dx}$ is also $\parallel \vec{\mathbf{A}}$.

In the case on the right, the vector $\vec{\mathbf{A}}'$ has the same *length* (A) as $\vec{\mathbf{A}}$ but is *not* in the same direction. The difference $\vec{\mathbf{\Delta A}} \equiv \vec{\mathbf{A}}' - \vec{\mathbf{A}}$ formed by the "tip-to-tip" rule of vector subtraction is also no longer in the same direction as $\vec{\mathbf{A}}$. In fact, it is useful to note that for these conditions (constant magnitude A), as the difference $\vec{\mathbf{\Delta A}}$ becomes *infinitesimally small* it also becomes *perpendicular* to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{A}}'$.[3] Thus the *rate of change* $\frac{d\vec{\mathbf{A}}}{dx}$ of a **vector** $\vec{\mathbf{A}}$ *whose magnitude A is constant* will always be *perpendicular to the vector itself*: $\frac{d\vec{\mathbf{A}}}{dx} \perp \vec{\mathbf{A}}$.

Centripetal Acceleration

From Fig. 10.4 we can see the relationship between the change in position $\Delta\vec{\mathbf{r}}$ and the change in velocity $\Delta\vec{\mathbf{v}}$ in a short time interval Δt. As all three get smaller and smaller, $\Delta\vec{\mathbf{v}}$ gets to be more and more exactly in the *centripetal direction* (along $-\hat{\mathbf{r}}$) and its scalar magnitude Δv will

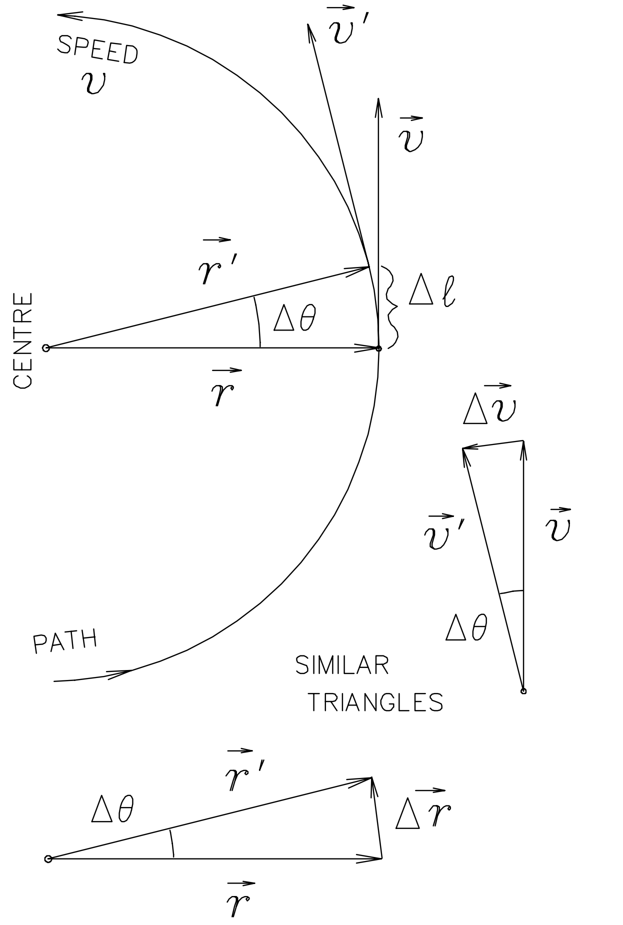

Figure 10.4 Differences between vectors at slightly different times for a body in uniform circular motion.

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always (from *similar triangles*) be given by

$$\frac{|\Delta\vec{\mathbf{v}}|}{v} = \frac{|\Delta\vec{\mathbf{r}}|}{r}$$

where I have been careful to write $|\Delta\vec{\mathbf{r}}|$ rather than Δr since the *magnitude* of the radius vector, r, does not change! Now is a good time to note that, for a tiny sliver of a circle, there is a vanishingly small difference between $|\Delta\vec{\mathbf{r}}|$ and the actual distance $\Delta\ell$ travelled along the arc, which is given exactly by $\Delta\ell = r\Delta\theta$. Thus

$$\frac{\Delta\vec{\mathbf{v}}}{v} \approx -\hat{\mathbf{r}}\,\frac{r\Delta\theta}{r} = -\hat{\mathbf{r}}\,\Delta\theta.$$

If we divide both sides by Δt and then take the limit as $\Delta t \to 0$, the approximation becomes

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[2] We write this $\vec{\mathbf{\Delta A}} \parallel \vec{\mathbf{A}} \parallel \vec{\mathbf{A}}'$ in standard notation.

[3] We write this $\vec{\mathbf{\Delta A}} \perp \vec{\mathbf{A}}$ in standard notation.

arbitrarily good and we get

$$\frac{1}{v}\left(\frac{d\vec{\mathbf{v}}}{dt}\right) = -\hat{\mathbf{r}}\left(\frac{d\theta}{dt}\right).$$

We can now combine this with the definitions of acceleration ($\vec{\mathbf{a}} \equiv d\vec{\mathbf{v}}/dt$) and angular velocity ($\omega \equiv d\theta/dt$) to give (after multiplying both sides by v) $\vec{\mathbf{a}} = -\hat{\mathbf{r}}\ \omega\ v$. We need only divide the equation $\Delta\ell = r\Delta\theta$ by Δt and let $\Delta t \to 0$ to realize that $v = r\omega$. If we substitute this result into our equation for the acceleration, it becomes

$$\vec{\mathbf{a}} = -\hat{\mathbf{r}}\,\frac{v^2}{r} = -\vec{\mathbf{r}}\,\omega^2 \qquad (2)$$

which is our familiar result for the *centripetal acceleration* in explicitly vectorial form.

Kepler

Empiricism

We are often led to believe that new theories are derived in order to explain fresh data. In actuality this is never the case. Theories are proposed to explain *experimental results*, which are always reported in an intermediate state of digestion somewhere between the raw data and the general explanatory theory. *Data* are merely meaningless bits of information and are often disregarded entirely unless and until their custodian (usually the Experimenter who collected them) translates them into some *empirical shorthand* that allows their *essential features* to be easily appreciated by other people. This is not always a simple task. Kepler, for instance, accumulated a large body of information in the form of observations of the positions of planets and stars as a function of time. In that form the data were incomprehensible to anyone, including Kepler. First he had to extract the *interesting part*, namely the positions of the planets *relative to the Sun*, from raw data complicated by the *uninteresting* effects of the Earth's rotation and its own annual trip around Sol, which required both a good model of what was basically going on and a *lot* of difficult calculations. Then, with these "reduced" data in hand, he had to draw pictures, plot different combinations of the variables against each other, and generally mull over the data (presumably scratching his head and thinking, "Now what the hell does *this* mean?" or his contemporary equivalent) until he began to notice some interesting *empirical generalizations* that could be made about his results. Of course I don't know exactly how Kepler went about this, but I do know the experience of turning new data over and over in my mind and on paper until some consistent empirical relationship between the variables "leaps out at me." And I am *very* impressed with the depth and delicacy of Kepler's observations.

Note that the Empiricist[4] has not *explained* the observed behaviour at this point, merely *described* it.[5] But a good description goes a long way! One should never underestimate the importance of this intermediate step in experimental science.

Kepler's Laws of Planetary Motion

1. Elliptical Orbits: The orbits of the planets are[6] *ellipses*[7] with the Sun at one of the foci.

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[4](who may or may not be the same person as the Experimentalist and/or the Theoretician — these are just different "hats" that a Physicist may put on)

[5]Of course, as in Kepler's case, the empirical description is always *in terms of* some preselected *model* or paradigm; but the paradigm in question is generally a familiar and widely accepted one, otherwise it is not very helpful in communicating the results to others. Besides, *the data themselves* are "collected" within the context of the Experimenter's paradigms and models about the world. The "simple" act of vision employs an enormous amount of "processing" in the visual cortex, as discussed earlier. . . .

[6](neglecting perturbations from the other planets, as is assumed in all Kepler's laws)

[7]Note that a *circle* is just a special case of an ellipse in which the major and semimajor axes are both equal to the *radius* and both *foci* are at the centre of the circle.

2. Constant Areal Velocity: The *area* swept out *per unit time* by a line joining the Sun to the planet in question is constant throughout the orbit.[8]

3. Scaling of Periods: The *square* of the *period* T of the orbit is proportional to the *cube* of the length of the semi-major axis (or, in the case of a circular orbit, the *radius* r) of the orbit:

$$T^2 \propto r^3$$

The Universal Law of Gravitation

By a process of logic that I will not attempt to describe, Newton deduced that the force F between two objects with masses m and M separated by a distance r was given by

$$F = \frac{GmM}{r^2} \tag{3}$$

where
$G = (6.67259 \pm 0.00085) \times 10^{-11}$ m^3·kg^{-1}·s^{-2} is the *Universal Gravitational Constant*. Actually, Newton didn't know the value of G; he only postulated that it was *universal* — *i.e.* that it was the same constant of proportionality for *every* pair of masses in this universe. The actual determination of the value of G was first done by Cavendish in an experiment to be described below.

We should also express this equation in vector form to emphasize that the force on either mass acts in the direction of the other mass: if $\vec{\mathbf{F}}_{12}$ denotes the force acting on mass m_2 due to its gravitational attraction by mass m_1 then

$$\vec{\mathbf{F}}_{12} = -\frac{G\, m_1\, m_2}{r_{12}^2}\, \hat{\mathbf{r}}_{12} \tag{4}$$

where $\hat{\mathbf{r}}_{12}$ is a unit vector in the direction of $\vec{\mathbf{r}}_{12}$, the vector distance from m_1 to m_2, and r_{12} is the scalar magnitude of $\vec{\mathbf{r}}_{12}$. Note that the reaction force $\vec{\mathbf{F}}_{21}$ on m_1 due to m_2 is obtained by interchanging the labels "1" and "2" which ensures that it is equal and opposite because $\vec{\mathbf{r}}_{21} \equiv -\vec{\mathbf{r}}_{12}$ by definition.

Weighing the Earth

Suppose you know your own mass m, determined not from your weight but from experiments in which you are accelerated horizontally by known forces. Then from your weight W you can calculate the mass of the Earth, M_E, if only you know G, the universal gravitational constant, and R_E, the radius of the Earth. The trouble is, you cannot use the same measurement (or any other combination of measurements of the weights of objects) to determine G. So how do we know G? If we can measure G then we can use our own weight-to-mass ratio (*i.e.* the acceleration of gravity, g) with the known value of $R_E = 6.37 \times 10^6$ m to determine M_E. How do we do it?

The trick is to measure the gravitational attraction between two masses m_1 and m_2 that are *both* known. This seems simple enough in principle; the problem is that the attractive force between two "laboratory-sized" masses is *incedibly tiny*.[9] Cavendish devised a clever method of measuring such tiny forces: He hung a "dumbbell" arrangement (two large spherical masses on opposite ends of a bar) from the ceiling by a long

[8] This feature, unlike the other two Laws, is true for *any* "central force" (a force attracting the body back toward the centre, in this case the sun). The other two are only true for *inverse square laws*, $F \propto 1/r^2$.

[9] If the Earth attracts a 1 kg mass with a force of 9.81 N, the gravitational force between two 1 kg masses separated by R_E would be smaller by a factor equal to the number of kilograms in M_E, which is a *large* number. Fortunately the smaller masses can be placed much closer together; this helps quite a bit, but the force is still miniscule!

thin wire and let the system come completely to rest. Then he brought another large spherical mass up close to each of the end masses so that the gravitational attraction acted to *twist the wire*. By careful tests on shorter sections of the same wire he was able to determine the *torsional spring constant* of the wire and thus translate the angle of twist into a torque, which in turn he divided by the moment arm (half the length of the dumbbell) to obtain the force of gravity F between the two laboratory masses M_1 and m_2 for a given separation r between them. From this he determined G and from that, using

$$g = \frac{G\,M_E}{R_E^2} \tag{5}$$

he determined $M_E = 5.965\times10^{24}$ kg for the first time. We now know G a bit better (see above) but it is a hard thing to measure accurately!

Orbital Mechanics

Let's pretend for now that all orbits are simple circles. In that case we can easily calculate the orbital radius r at which the centripetal force of gravitational attraction F is just right to produce the centripetal acceleration a required to maintain a steady circular orbit at a given speed v. For starters we will refer to a light object (like a communication satellite) in orbit about the Earth.

Orbital Speed

The force and the acceleration are both *centripetal* (*i.e.* back towards the centre of the Earth, so we can just talk about the *magnitudes* of $\vec{\mathbf{F}}$ and $\vec{\mathbf{a}}$:

$$F = \frac{GmM_E}{r^2} \qquad \text{and} \qquad a = \frac{v^2}{r}.$$

$$\text{but} \qquad F = ma, \qquad \text{so}$$

$$\frac{GmM_E}{r^2} = \frac{mv^2}{r} \Longrightarrow \frac{G\,M_E}{r} = v^2.$$

We can "solve" this equation for v in terms of r,

$$v = \sqrt{\frac{G\,M_E}{r}}, \tag{6}$$

or for r in terms of v:

$$r = \frac{G\,M_E}{v^2}. \tag{7}$$

You can try your hand with these equations. See if you can show that the orbital velocity at the Earth's surface (*i.e.* the speed required for a frictionless train moving through an Equatorial tunnel to be in free fall all the way around the Earth) is 7.905 km/s. For a more practical example, try calculating the radius and velocity of a *geosynchronous satellite* — *i.e.* a signal-relaying satellite in an Equatorial orbit with a period of exactly one day, so that it *appears* to stay at exactly the same place in the sky all the time.[10]

Changing Orbits

The first thing you should notice about the above equations is that satellites move *slower* in *higher orbits*. This is slightly counterintuitive in that they go slower when they have further to go to get all the way around, which has a dramatic effect on the period (see below). However, that's the way it is. Consequently, if you are in a *low* orbit and you want to transfer into a *higher* orbit, you eventually want to end up going *slower*. Nevertheless, the first thing you do to initiate such a change is to *speed up*! See if you can figure out why.[11]

Periods of Orbits

We can now explain (at least for circular orbits) Kepler's Third Law. The period T of an orbit

[10] If you have a TV satellite dish, it is pointing at such a satellite; note that (if you live in the Northern Hemisphere) it is tipped toward the South. Why?

[11] (The most intuitive explanation for this involves the concepts of *kinetic and potential energy*, which we will watch *emerge* from Newton's Mechanics in succeeding Chapters.

is the circumference $2\pi r$ divided by the speed of travel, v. Using the equation above for v in terms of r gives

$$T = \frac{2\pi r}{\sqrt{\frac{GM_E}{r}}}$$
$$= \frac{2\pi}{\sqrt{GM_E}} r^{\frac{3}{2}}$$
$$\text{or} \qquad T^2 \propto r^3$$

as observed by Kepler. Newton explained *why*.

Tides

Here on the surface of the Earth, we have little occasion to notice that the force of gravity drops off inversely as the square of the distance from the centre of the Earth.[12] This is fortunate, since otherwise Galileo would not have been able to do his experiments demonstrating the (approximate) constancy of the acceleration of gravity, g; moreover, scales and other mass-measuring technology based on uniform gravity would not work well enough for commerce of engineering to have evolved as it did. So we don't notice any effects of the inverse square law "here at home," right? Well, let's not be hasty.

The Moon exerts an infinitesimal force on every bit of mass on Earth. At a distance of $R_ME = 380,000$ km, the Moon's mass of $M_M = 7.4 \times 10^{22}$ kg generates a gravitational acceleration of only $g_ME = 3.42 \times 10^{-5}$ m/s^2; in other words, our gravitational attraction to the Moon is 3.5×10^{-6} of our Earth weight. Moreover, the Moon's gravitational acceleration *changes* by only -1.8×10^{-13} m/s^2 for every metre further away from the Moon we move — a really *tiny* gravitational *gradient*. Nevertheless, the fact that the water in the oceans on the side of the Earth facing the Moon is attracted *more* and that on the side away from the Moon is attracted *less* leads to a slight *bulge* of the water on *both sides* and a concomitant *dip* around the middle. As the Earth turns under these bulges and dips, we experience (normally) *two* high tides and *two* low tides every day.

The consequences of these tides are nontrivial, as we all know. Even though they are the result of an incredibly small gravitational gradient, they represent enormous energies that have been tapped for power generation in a few places like the Bay of Fundy where resonance effects generate huge movements of water. More importantly in the long run (but of negligible concern in times of interest to humans) is the fact that the "friction" generated by these tides is gradually sapping the kinetic energy of the Earth's rotation and at the same time causing the Moon to drift slowly further away from the Earth so that in a few billion years the Earth will be "locked" as the Moon is now, with its day the same length as a month (which will then be twice as long as it is now) and the same side always facing its partner. *"Sic transit gloria Mundi,"* indeed! Let's enjoy our spin while we can.

A less potent source of tidal forces (gravitational gradients) on Earth is the Sun, with a mass of about 3×10^{40} kg at a distance of about 93 million miles or 1.5×10^{11} m. You can calculate for yourself the Sun's gravitational acceleration at the Earth: small but not entirely negligible. The Sun's gravitational *gradient*, on the other hand, is truly miniscule; yet various species of fish seem to have feeding patterns locked to the relative positions of the Sun and the Moon, even at night when the more obvious effects of the Sun are absent. The so-called "solunar tables" are an essential aid to the fanatically determined

[12] Surely by now you have gotten skeptical of my repeated declarations that the mass of the Earth can be treated as if it were all concentrated at the Earth's centre of gravity (*i.e.* the centre of the Earth). What about all the bits right next to us? They have a much smaller r^2 and thus contribute far more "pull" than those 'way on the other side. Well, hang on to that skepticism! I'm not leading you astray (promise!) but a little later on I will be in a better position to use *Gauss' Law* to explain in a few quick steps why this works. You should only provisionally accept this notion until you have seen a convincing argument with your own eyes.

fisherman! Yet, so far as I know, no one has any plausible explanation for how a fish (or a bird or a shellfish, which also seem to know) can detect these minute force gradients.

A more dramatic example of tidal forces is the gravitational field near a *neutron star*, which has a large enough gradient to dismember travellers passing nearby even though their orbits take them safely past.[13] Near a *small black hole* the tidal forces can literally rip the vacuum apart into matter and antimatter, causing the black hole to explode with unmatched violence; this in fact limits how small black holes can be and still remain stable.[14]

[13] This *motif* has been used in several delightful science fiction stories, notably "Neutron Star" by Larry Niven. and ? Egg ? by ? .

[14] Bill Unruh, of the UBC Physics Department, is one of the world's leading experts on this subject.