FALLING BODIES

Now that we have mastered all sorts of Algebra and Calculus skills, it is time to get on with Newtonian Mechanics, Gravitation, Cosmology and all that, right?

Gee, I sure wish it were true.

Although (I hope you will agree) there are some interesting historical and perceptual lessons to be learned from Newton's Mechanics, it is generally rated as one of the more boring topics in Physics;[1] worse yet, we are not yet ready for Newton — "You have to creep before you can crawl," as it were. And in this business "creeping" is the business of *Kinematics* — the study of *motion per se*.

Besides, before we go on to expound Newton's "Laws" in their modern form we will need to have a chapter on *vectors*, since *forces* are classic examples of vectors — *i.e.* they have both *magnitude* and *direction*.[2]

6.1 Galileo

As I warned the reader in several places earlier, I am no historian. However, I do have many traits in common with real Historians; in particular, I like to construct theories of "what probably *really* happened" to fit my own interpretation of the historical "data." Physicists also like this sort of revisionism, but I think we are mercifully more shameless and direct about it. ["Yeah, OK, I lied; but it was a *good* lie — doesn't it make everything easier to understand?"] With this *caveat*, I will relate a bit of Brewer's History of Classical Mechanics.

Galileo Galilei (1564-1642) was a clever Italian megalomaniac who took pleasure in publicly ridiculing his intellectual opponents and regarded the authorities as annoying buffoons to be manipulated by any means available in order to obtain funding for his pet projects. He thus epitomized a fine tradition which continues to this day. Galileo is widely credited with being "the Father of Modern Science" because of the experimental æsthetics he championed[3] and because of the impact of his major work, *Two New Sciences* [mechanics and the strength of materials], published in 1636. I am inclined to think that his distinctive personality and style had just as much to do with his deserving this title; today these traits are still apt to improve the bearer's chances for distinction by various prizes and accolades.

6.1.1 Harvard?

Rather than reproduce the list of Galileo's adventures available in any textbook with even a pretense of historical perspective, I will mention one amusing claim that I heard somewhere:[4] when Galileo got into trouble with the Church over his heretical views[5] he was offered a faculty position at a new University in another country where the Roman Church was not all that

[1] This is partly because everyone is so anxious to "get on to the good stuff" that they are predisposed to give a rather superficial treatment to Mechanics; and partly because most beginning Physics courses are expected to produce graduates who can actually calculate tensions in wires, whether boxes will slide off trucks and other practical things like that. Fortunately, I don't care whether you can do that stuff or not, except for a few simple examples for the sake of illustration and familiarization. This book *may* help you build a bridge in your back yard, but honestly I think there are much more useful study aids for developing such skills. What I am after is just to get you familiar enough with the *paradigms* of Mechanics to allow bootstrapping on to the next stage.

[2] This is also true of *distance*, *velocity* and *acceleration*, which are the topics of this Chapter; but we have to start *somewhere*.

[3] Often referred to as the "Scientific Method," about which I will have more to say later on.

[4] You real Historians go check this out!

[5] Actually they would probably have left him alone if he hadn't been so obnoxious about publicly rubbing their noses in it.

popular — the school in question was Harvard.[6]

6.1.2 Weapons Research: Telescopes and Trajectories

Ever the Modern Physicist, Galileo recognized clearly that the big money and prestige were in military applications of science. In those days the new weapons technology was *cannons* and how to aim them more accurately at targets. His contributions to this art took two main forms: the first was his invention of the *magnifying telescope*, with which it was possible to identify targets at great range and assess the damage done to them by one's cannonballs. To be fair, I should point out that this invention was warmly received by seafarers and astronomers as well as generals; in fact, with it Galileo himself made famous and wonderful observations of the Moon, the "Galilean" moons (named after guess whom) of Jupiter and numerous other objects in our Solar System, thereby initiating the modern pastime of Planetology that recently culminated in the fantastic close-up views of the outer planets and their satellites by Terran space probes. One can easily imagine how ridiculous the Church's Ptolemaic ergocentric model of the Heavens must have seemed to Galileo after watching so many other planets execute their orbits as clearly visible globes lit on the Sun side.[7] There are two sides to every coin.

Galileo's second contribution to the art of artilliery was his formal explication of the behaviour of *falling bodies*, of which cannon and musket balls were oft-mentioned examples. Galileo "showed"[8] that the velocity of a falling body increases by equal increments in equal times (in the absence of friction), which is the definition of a state of constant acceleration.

Constant Acceleration

In terms of our newly-acquired left hemisphere skills, if we use y to designate *height* [say, above sea level] and t to designate *time*, then the *upward velocity* v_y [where the subscript tells us explicitly that this is the *upward* velocity as opposed to the *horizontal* velocity which would probably be written v_x][9] is given by

$$v_y = v_{y_0} - gt \tag{1}$$

where v_{y_0} is the initial[10] upward velocity (*i.e.* the upward velocity at $t = 0$), if any,[11] and g

[6] One imagines Galileo's response was, "I'm not *that* desparate." In those days Harvard had presumably not yet acquired much of a reputation. It is amusing to speculate on how much *more* classic an example of the Modern Physicist he would have made had Galileo accepted this offer of a New World professorship.

[7] The astronomical observations of Tycho Brahe and Johannes Kepler empirically obliterated the Ptolemaic system in favour of a correct heliocentric model of the Solar system at about the same time as Galileo took on the Church in Italy; I am not certain how much interaction there was between these apparently separate battles. More on this later.

[8] There is room for argument over whether he really "showed" this, both from a Popperian purist's point of view [you can never *verify* a conjecture, only *refute* it] and from the point of view of the very æsthetic he helped to popularize — namely, that you shouldn't "fudge" your results and that other people should be able to *reproduce* them. It is, however, certainly true that he *made a very persuasive case* for the economy and utility of this confessed overidealization; and this is, after all, the true measure of any theory!

[9] Why not just call it v, if I am not going to be talking about any of the horizontal stuff? Well, this is a pretty simple equation, so I am going to "stack" it with lessons in *notation* which will serve to make its meaning absolutely unambiguous (subject to all these explanations) and to introduce fine points I will be needing shortly anyway.

[10] Note: generally any symbol with a subscript $_0$ (read "nought" as in x_0 = "x nought") designates an *initial value* of the subscripted symbol — *i.e.* the value at $t = 0$. (We stop short of writing t_0 for the initial time, in most cases, because we usually don't need any further redundancy to make the the description completely general.) Thus x may be a variable, a function of time $x(t)$, but its initial value $x_0 \equiv x(0)$ is a constant, a *parameter* of its evolution in time. Since we will often talk about the *final* value of some variable (*e.g.* x_f) at time t_f (at the end of some process), using the subscript $_f$ to designate "final," it is equally logical to use a subscript $_i$ for "initial," so that the value of $x(t)$ at $t = 0$ would be written x_i — this notation is *perfectly synonymous* with the "nought" notation: $x_0 \equiv x_i$ and the two may be used interchangably according to taste.

[11] Lots of people leave out the v_{y0} in order to keep it simpler, but of course that would be tantamount to assuming that we were starting *from rest*, which ain't necesarily so! Why *over*simplify an already simple equation?

is the downward[12] acceleration of gravity, $g \approx 9.81\ \mathrm{m/s^2}$ on average at the Earth's surface.[13] Another way of writing the same equation is in terms of the *derivative* of the velocity with respect to time,

$$a_y \equiv \frac{dv_y}{dt} \equiv \dot{v_y} = -g, \tag{2}$$

where I have introduced yet *another* notational convention used by Physicists: a little dot above a symbol means the *time derivative* of that symbol — *i.e.* the rate of change (per unit time) of the quantity represented by that symbol.[14] And since v_y is itself the time derivative of the height y [*i.e.* $v_y \equiv dy/dt \equiv \dot{y}$], if we like we can write the original equation as

$$\dot{y} = v_{y_0} - gt. \tag{3}$$

All these notational gymnastics have several purposes, one of which is to make you appreciate the simple clarity of the declaration, "The vertical speed increases by equal increments in equal times," as originally stated by Galileo himself. But I also want you to see how Physicists like to *condense* their notation until a very compact equation "says it all."

[12]Note that the conventional choice of "up" as being the *positive* y direction forces us to put the acceleration of gravity into the equation with a minus sign, since it is in the "down" direction. Sometimes people try to make this look simpler for beginners by defining *down* as the $+y$ direction, but I like to get across as early as possible that a negative acceleration simply means an acceleration in the direction opposite to the one we arbitrarily defined to be positive. The same is true of any quantity (*e.g.* the velocity or the position) that has a direction as well as a magnitude; this idea is vital to an understanding of *vectors*, which are coming up soon!

[13]What?! How come I don't give g to a huge number of significant figures, with an uncertainty specified, as one is supposed to do for fundamental constants? Because g is neither fundamental nor constant! Far from it. More on this later.

[14]I will soon need the analogous notation $\ddot{x} \equiv d^2x/dt^2$ to signify the *second* time derivative of x, so that $a_y \equiv dv_y/dt \equiv d^2y/dt^2 \equiv \ddot{y}$. The "double-dot" form is the preferred Physics notation for acceleration, mainly for reasons of economy (it takes so few strokes to write).

The Principles of Inertia and Superposition

Galileo was actually the first to write down "Newton's" celebrated First Law, in a form slightly different from Newton's but just as good:[15]

Galileo's Principle of Inertia:

> *A body moving on a level surface will continue in the same direction at constant speed unless disturbed.*

Note the term "body" employed in order to be deliberately vague about what *sort* of entities the Principle is meant to apply to. This term is retained in the language of modern Mechanics. It means, more or less, "a massive thing that hangs together." Note also the other ringers, "level surface" and "unless disturbed." *Perfectly* level surfaces are mighty hard to come by, but Galileo means, of course, a *hypothetical* perfectly level surface. More serious is the vagueness of "unless disturbed." This can easily be used to make the argument circular: if the body's velocity changes direction or magnitude, it is because it is "disturbed." Well. . . . Newton invented a new concept to make "disturbance" a little more specific.

The other important insight Galileo saw fit to enshrine as a Principle was

Galileo's Principle of Superposition:

> *If a body is subjected to two separate influences, each producing a characteristic type of motion, it responds to each without modifying its response to the other.*

[15]This is translated from the Italian by someone else; I can't vouch for the translation but I am confident that it gets the right idea across and I am not much interested in quibbles over the exact wording or what it might have meant about Galileo's "authentic originality."

This, like the other Principle, seems transparently obvious to Modern eyes,[16] but without it one would never know how to start applying Galileo's simplified kinematics to the practical problem of trajectories. Again there is a little sloppiness to the Principle that allows for counterexamples; no doubt Galileo had to rely regularly on the most honest of all appeals: "*You* know what I mean."

Calculating Trajectories

Applied to the case of *trajectories* close to the Earth's surface,[17] the equations governing constant horizontal velocity *superimposed* upon constant downward acceleration take the form

$$\begin{aligned} \ddot{x} &= 0 && (4)\\ \dot{x} &= v_{x_0} \quad \text{(constant)} && (5)\\ x &= x_0 + v_{x_0}\, t && (6)\\ \text{and} & && (7)\\ \ddot{y} &= -g && (8)\\ \dot{y} &= v_{y_0} - g\, t && (9)\\ y &= y_0 + v_{y_0}\, t - \frac{1}{2} g\, t^2 && (10) \end{aligned}$$

where

$$\begin{aligned} \ddot{x} &\equiv \frac{d^2x}{dt^2} \equiv \frac{dv_x}{dt} \equiv \dot{v_x} \equiv a_x, && (11)\\ \dot{x} &\equiv \frac{dx}{dt} \equiv v_x, && (12)\\ \ddot{y} &\equiv \frac{d^2y}{dt^2} \equiv \frac{dv_y}{dt} \equiv \dot{v_y} \equiv a_y && (13)\\ \text{and} \quad \dot{y} &\equiv \frac{dy}{dt} \equiv v_y && (14) \end{aligned}$$

Hold it! Before you bolt for the door, take a moment to casually read through all these horrible-looking equations. I have made them look long and hirsute *on purpose*, for two reasons: first, because this way they are in their *most general form* — *i.e.* we can be confident that these equations will correctly describe *any* trajectory problem, but for any actual problem the equations will usually *simplify*; and second, because this is a sort of practical joke — if you look carefully you will see that the equations are really pretty simple! All those "$\equiv$" symbols just mean, "...another way of putting it, which amounts to exactly the same thing, is...." That is, they just indicate *equivalent notations* — or, in the language of linguistics, *synonyms*. So the latter batch of equations is just reminding you of the convention Physicists use for writing time derivatives: "dot" and "double-dot" notation. The first batch of equations tells you (in this notation) *everything there is to know* about the motion: the horizontal $[x]$ motion is not under any acceleration $[a_x \equiv \ddot{x} = 0]$ so the horizontal velocity $[v_x \equiv \dot{x}]$ is constant $[\dot{x} = v_{x_0}]$ and the distance travelled horizontally $[x(t)]$ is just increasing linearly with time t relative to its initial value x_0 — *i.e.* $x = x_0 + v_{x_0}t$. The vertical motion differs only in that it includes a constant downward acceleration $[a_y \equiv \ddot{y} = -g]$ which adds a term $[-gt]$ to $\dot{y}$ and another familiar term $[-\frac{1}{2}gt^2]$ to $y(t)$. Note that in every case the whole idea is to get the quantity on the left-hand side $[lhs]$ of the equation equal to an *explicit function of* t on the right-hand side $[rhs]$.

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[16] This may well be a good measure of the brilliance of an insight.

[17] *E.g.*, cannonballs! This sort of "techno doubletalk" is not *always* used for obfuscation [I, for instance, am simply trying to be general!] but Pentagon aides trying to be Generals are very fond of it too.

Let's do a problem to illustrate how these equations work: Suppose we fire a cannon horizontally from the top of a 19.62 m high bluff, imparting an initial velocity $v_{x_0} = 10$ m/s to the cannonball. [By the definition of "horizontal," $v_{y_0} = 0$.] Where does the ball hit? [We neglect air friction and assume level (horizontal) ground at the bottom of the bluff.] For simplicity we can take $x = 0$ at

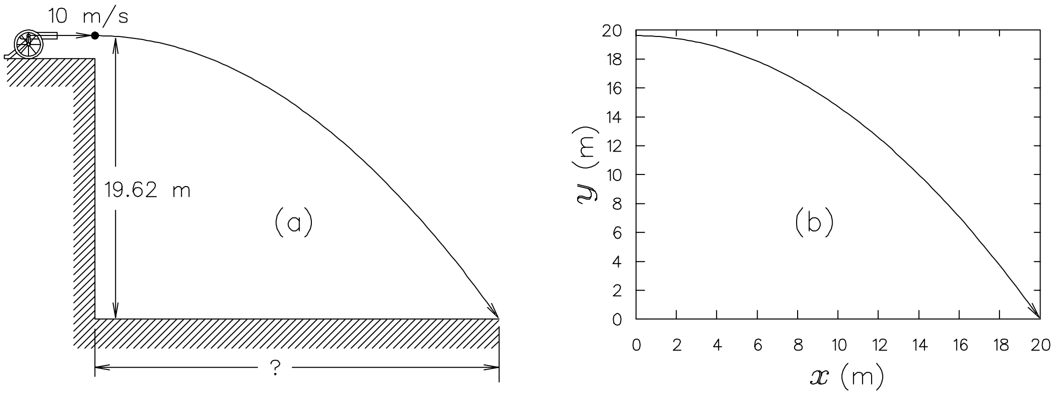

Figure 6.1 (a) Sketch of a trajectory problem in which the initial height [$y_0 = 19.62$ m] and the initial (horizontal) velocity [$v_{x_0} = 10$ m/s] are given and we want to calculate the horizontal distance [x_f] at which the cannonball hits the ground [$y_f = 0$]. (b) Corresponding plot of $y(x)$, the *trajectory* followed by the cannonball.

the muzzle of the cannon;[18] similarly, we (naturally enough) take $t = 0$ to be the instant at which the ball leaves the muzzle of the cannon. Our *general* equations now "reduce" to a more *particular* set of equations for this specific example:

$$x = v_{x_0}\, t \qquad \text{and} \qquad y = y_0 \;-\; \frac{1}{2}\, g\, t^2$$

or, since $v_{x_0} = 10$ m/s and $y_0 = 19.62$ m,

$$x = (10\text{m/s})\, t \qquad \text{and} \qquad y = (19.62\text{m}) \;-\; \frac{1}{2}\, (9.81\text{m/s}^2)\, t^2$$

We now have a choice between working out the algebra in the first pair of equations or working out the arithmetic in the second pair. The former is preferable partly because we don't have to "juggle units" while we work out the equations (a clumsy process which is usually neglected, leading to equations with numbers but no units, which in turn can lead to considerable confusion) and because solving for x_f in terms of the two "parameters" y_0 and v_{x_0} [g is also a parameter, although we usually treat it as if it were a *constant* of Nature] gives an "answer" to *any such problem* with *qualitatively* similar conditions. Here's the algebra:

$$x = v_{x_0}\, t \qquad \Longrightarrow \qquad t = \frac{x}{v_{x_0}}$$

which can be substituted for t in the second equation, giving

$$y = y_0 \;-\; \frac{1}{2}\, g \left[\frac{x}{v_{x_0}} \right]^2 .$$

[18](This is typical — we always make as many simplifications as the arbitrariness of the notation allows!)

We are interested in the value of x_f at the end of the trajectory — *i.e.* when $y_f = 0$:

$$y_f = 0 = y_0 - \frac{1}{2} g \left[\frac{x_f}{v_{x0}} \right]^2 \qquad \Longrightarrow \qquad y_0 = \frac{1}{2} g \frac{x_f^2}{v_{x0}^2}$$

$$\Longrightarrow \qquad \frac{2y_0}{g} = \frac{x_f^2}{v_{x0}^2} \qquad \Longrightarrow \qquad \frac{2y_0\, v_{x0}^2}{g} = x_f^2 \qquad \Longrightarrow \qquad x_f = \sqrt{\frac{2y_0\, v_{x0}^2}{g}}.$$

Now we "plug in" $y_0 = 19.62$ m, $v_{x_0} = 10$ m/s and $g = 9.81$ m/s^2, giving

$$x_f = \sqrt{\frac{2 \times 19.62\text{m} \times [10\text{m/s}]^2}{9.81\text{m/s}^2}} = \sqrt{\frac{2 \times 2 \times 9.81 \times 100\text{m}^3/\text{s}^2}{9.81\text{m/s}^2}} = \sqrt{400\text{m}^2} = 20\text{m}.$$

And that's the answer: $x_f = 20$ m. Simple, huh?

6.2 The Scientific Method

One often hears that "the modern Scientific Method" can be traced back to Galileo, who first prescribed the panacea of "Observe, Hypothesize, Experiment and Confirm." This is complete nonsense.[19]

First of all, people have been doing more or less the same thing since before the Dawn of Recorded History;[20] Galileo just grabbed the headlines when there was first Good Press to get! He was a hero, true, in that he championed the arrogance of *thinking for oneself* against formidable odds and outlined a procedure for doing it successfully (*i.e.* getting away with it) for which we all are in his debt. But he could hardly claim a patent on the idea.

Second, Galileo's Scientific Method, like his Mechanics, was an *idealization* of an imperfect experimental reality. As discussed earlier, we cannot Observe without relying upon our *repertoire* of *models* through which we interpret our sense data; the phrase, "Seeing is believing," betrays a profound *naiveté* if we consider carefully what we know about the retina, the optic nerve and the visual cortex. We may Hypothesize freely, but only the most righteous scientists are actually honest about *when* their hypotheses were formed — before or after the experiment![21] The one part of Galileo's prescription that we truly took to heart was the exhortation to *Experiment* — *i.e.* to go directly to Nature with our questions about "what will happen if we...?" Asking such questions in a form that Nature will deign to answer unambiguously is a profound art indeed; a lifetime is too short to learn it in. Finally, Galileo can be considered charmingly naive in his expectation that Experimentation will be able to Confirm any Hypothesis. As Karl Popper has pointed out, there is no logical basis upon which any "general explanatory theory" can be proven correct by any finite number of experiments; the best we can hope for is a Conjecture which is "not yet Refuted" by the evidence, and this is impressive only if there is a *lot* of non-contradictory evidence!

[19] By now, you no longer need to be reminded that such comments are "in my humble opinion."

[20] Why not the *Sunset* of Recorded History, I sometimes wonder?

[21] Newton, whom we often picture as the gardener who brought Galileo's seeds to flower, is also famous for his arrogant statement [a blatant lie], "*Hypotheses non fingo*," or "I do not make conjectures." (What a jerk!)

So the revised version of the "Scientific Method" should read something like this:

1. Based on a lifetime of experience, form a Hunch.

2. Using a trained analytical mind, refine the Hunch into a well-posed Hypothesis.[22]

3. Think of a few Consequences of the Hypothesis that lead to Predictions that can be tested by Experiment.[23]

4. Perform a *Gedankenexperiment*[24] to visualize the results you should expect to get under different circumstances.

5. Design a real Experiment, if possible, to produce the most clear and unambiguous results[25] possible.

6. Descend to the level of grubby sociopoliticoeconomic reality to seek funding, recruit personnel, fight battles for priority, coordinate with engineers, construct several versions of the apparatus (all but the last of which do not work), tinker with balky equipment, coax plausible results out of partially recorded data, argue with collaborators about procedure and interpretation, *etc.*, for as long as it takes to get the Experiment done [which may exceed your lifespan in certain disciplines].

7. Publish a Result (or Results) — often determined by "consensus" [*i.e.* politics] among Collaborators — and let the Community decide what it means.

8. Go back to Step 1, if you did not already do so earlier.

Of course, these are the rules for a Professional Scientist; if you are content to remain an Amateur, the Scientific Method is a little simpler:

$\boxed{\textit{Think for yourself.}}$

In all the above arguments, there is an implicit assumption that we usually do not discuss: namely, that there is an "external" Real World independent of our perceptions and models that behaves the way it does regardless of our expectations or observations — that we can, at least in spirit, set ourselves apart from The World as mere *observers* of its behaviour. Even in Classical Mechanics this is an obvious idealization, but perhaps a conscionable one. In Quantum Mechanics (as we shall see) this basic view of the Experimenter as Observer is challenged at its roots! Nevertheless there are things we can do which *seem like* Observations and which we will have to use to "pull ourselves up by the bootstraps" if we are to even grasp what Quantum Mechanics has to tell us. So, for the time being, I encourage you to steep yourself in the traditional æsthetic of Experimental Science and try to be as "objective" and "non-interfering" as possible in making (or imagining) your Experimental Observations.

6.3 The Perturbation Paradigm

Galileo "demonstrated" the phenomenon of constant acceleration using a water clock and a ball

[22] This is not as easy as it sounds. Most Hunches do not survive close examination; they usually contain irreducible internal inconsistencies or self-contradictions that may, at best, lead the Scientist back to a completely new Hunch.

[23] This is also harder than it sounds. Many Hypotheses have no testable Consequences at all; most of the rest could be tested in principle but might require manipulation of galaxies or reenactments of the Big Bang to produce unambiguous experimental results.

[24] *I.e.*, a "thought experiment." This term was invented by Albert Einstein, I believe, but the *technique* is as old as Humanity — this was the approved methodology of Aristotelian science, and is still a great boon to research funding agencies!

[25] *I.e.*, those most commensurate with conventional models and paradigms, either *pro* or *con* the Predictions of the Hypothesis.

rolling down an inclined groove. In my experience, even with modern equipment it is difficult to obtain decent data on this sort of phenomenon; and even these data are typically *not* consistent with a true state of constant acceleration! There is no doubt that Galileo was quite aware of these flaws in his description; he was also quite happy to consign them to the realm of the "non-ideal" — *i.e.* the deviations from his predictions were due to *imperfections* in the ramp and the *disturbance* of the motion by the presence of *air*. Galileo argued that the results of a falling-body experiment performed *underwater* would be a lot *worse* than those of his experiments in *air*, so that one merely needed to extrapolate to *no medium at all* (*i.e.* perfect vacuum) to obtain results in perfect agreement with his predictions!

This overtly Platonic idealism was not new; but Galileo had hit upon a "good" approximation — one which actually *did* work better and better as the circumstances got closer and closer to a well-defined ideal case. The corrections could be regarded as negligible *perturbations* upon an "essentially correct" idealization, to be beaten into submission either by improvement of the apparatus or by laborious calculations.

Thus began what I call the "Perturbation Paradigm" of Physics. This simple prescription — find a nice simple model that does "pretty well" and then "fix up" its inadequacies with a series of corrections or "perturbations" — is so powerful that we Physicists use it on almost everything. The recent history of elementary particle physics gives a particularly poignant example of how a problem that was seemingly intractable by this perturbative method (and which promised for a while to lead us into genuinely new ways of thinking, which might have been nice for a change) was finally recast into a form that allowed application of the Perturbation Paradigm after all. I will suppress the urge to tell you about it now. But just wait!