The Language of Math

Soon we will tackle the problem of *measurement*, with all its pitfalls and practical tricks. You may then sympathize with Newton, who took such delight in retreating into the Platonic ideal world of pure mathematics, where relationships between "variables" are not fraught with messy errors, but defined by simple and elegant prescriptions. No matter that we are unable to measure these perfect relationships directly; this is merely an unfortunate consequence of our imperfect instruments. (Hmmm....) But first we need to describe the notational conventions to be used in this book for the language of Mathematics, without which Physics would have remained mired in the rich but confusing ambiguities of natural language. Here is where we assemble the *symbols* into *structures* that express (in some conventional idiom) the *relationships* between the "things" the symbols represent.

Please do not feel insulted if the following review seems too elementary for someone at your level. I have always found it soothing to review material that I already know well, and am usually surprised to discover how much I forgot in such a short while. Also, I think you'll find it picks up a bit later on.

4.1 Arithmetic

We have already dwelt upon the formalism of Number Systems in a previous Chapter, where we reminded ourselves that just counting to ten on paper involves a rather sophisticated and elaborate representational scheme that we all learned as children and which is now *tacit* in our thought processes until we go to the trouble to dismantle it and consider possible alternatives.

Arithmetic is the basic algebra of Numbers and builds upon our tacit understanding of their conventional representation. However, it would be emphatically wrong to claim that, "Arithmetic is made up of Numbers, so there is nothing to Arithmetic but Numbers." Obviously Arithmetic treats a new *level* of understanding of the properties of (and the relationships between) Numbers — something like the Frank Lloyd Wright house that was not there *in* the bricks and mortar of which it is built. [One can argue that in fact the conceptual framework of Number Systems implicitly contains intimations of Arithmetic, but this is like arguing that the properties of atoms are implicit in the behaviour of electrons; let's leave that debate for later.]

We learn Arithmetic at two levels: the *actual* level ("If I have two apples and I get three more apples, then I have five apples, as long as nothing happens to the first two in the meantime.") and the *symbolic* level ("2+3=5"). The former level is of course both *concrete* (as in all the *examples*) and profoundly *abstract* in the sense that one learns to understand that two of anything added to three of the same sort of thing will make five of them, independent of words or numerical symbols. The latter level is more for *communication* (remember, we have to adopt and adapt to a notational convention in order to express our ideas to each other) and for *technology* — *i.e.* for developing *manipulative tricks* to use on Numbers.

Skipping over the simple Arithmetic I assume we all know tacitly, I will use *long division* as an example of the conventional technology of Arithmetic.[1] We all know (today) how to

[1] No doubt the useful lifetime of this example is only a few more years, since many students now learn to divide by punching the right buttons on a hand calculator, much to the dismay of their aged instructors. I am not so upset by this — one arithmetic manipulation technology is merely supplanting another — except that "long division" is *in principle* completely understood by its user, whereas few people have any idea what actually goes on inside an electronic calculator. This dependence on mysterious and unfamiliar technology may have unpleasant long-term psychological impact, perhaps making us all more willing to accept the judgements of authority figures without question.... But in Mathemat-

do long division. But can we *explain how it works*? Suppose you were Cultural Attaché to Alpha Centauri IV, where the local intelligent life forms were interested in Earth Math and had just mastered our ridiculous decimal notation. They understand addition, subtraction, multiplication and division perfectly and have developed the necessary skills in Earth-style gimmicks (carrying, *etc.*) for the first three, but they have no idea how we actually go about dividing one multi-digit number by another. Try to imagine how you would explain the long division trick. Probably by example, right? That's how most of us learn it. Our teacher works out *beaucoup* examples on the blackboard and then gives us *beaucoup* homework problems to work out ourselves, hopefully arrayed in a sequence that sort of leads us through the process of *induction* (not a part of Logic, according to Karl Popper, but an important part of human thinking nonetheless) to a bootstrap grasp on the method. Nowhere, in most cases, does anyone give us a full rigourous derivation of the method, yet we all have a deep confidence in its universality and reliability — which, I hasten to add, I'm sure *can* be rigourously derived if we take the trouble. Still, we are awfully trusting....

The point is, as Michael Polanyi has said, *"We know more than we can tell."* The *tacit* knowledge of Arithmetic that you possess represents an enormous store of

- sophisticated abstract understanding
- arbitrary conventions of representational notation
- manipulative technology

that have already coloured your thought processes in ways that neither you nor anyone else will ever be able to fathom. We are all brainwashed by our Grammar school teachers![2] This book, if it is of any use whatsoever, will have the same sort of effect: it will "warp" your thinking forever in ways that cannot be anticipated. So if you are worried about being "contaminated" by Scientism (or whatever you choose to label the paradigms of the scientific community) then stop reading immediately before it is too late! (While you're at it, there are a few other activities you will also have to give up....)

ics, as long as you have *once* satisfied yourself completely that some technology is indeed trustworthy and reliable, of course you should make use of it! (Do you *know* that your calculator *always* gives the right answers...?)

4.2 Geometry

In Grammar school we also learn to recognize (and learn the grammar of) geometrical shapes. Thus the Right Hemisphere also gets early training. Later on, in High School, we get a bit more insight into the *intrinsic* properties of Euclidean space (*i.e.* the "flat" kind we normally *seem* to be occupying).

4.2.1 Areas of Plane Figures

- The area A of a *square* is the *square* of the length ℓ of any one of its 4 sides: $A = \ell^2$. In fact the question of which word "square" is named after which is a sort of chicken *vs.* egg problem for which there is no logical resolution (even though there may be an historically correct etymological answer).
- The area A of a *rectangle* (a bit more general) is the product of the length b of a long side ("base") and the length h of a short side ("height"): $A = bh$.

[2]It occurs to me that Grammar school is called Grammar school because it is where we learn *grammar* — *i.e.* the *conventional representations* for things, ideas and the relationships between them, whether in verbal language, written language, mathematics, politics, science or social behaviour. These are usually called "rules" or even (when a particularly heavy-handed emphasis is desired) "laws" of notation or manipulation or behaviour. We also pick up a little *technology*, which in this context begins to look pretty innocuous!

- The area A of a *triangle* with base b and height h (measured from the opposite vertex down perpendicular to the base) is $A = \frac{1}{2}bh$. (This is easy to see for a *right* triangle, which is obviously half a rectangle, sliced down the diagonal. You may want to convice yourself that it is also true for "any old triangle.")
- The area A of a *circle* of radius r is given by $A = \pi r^2$ where π is a number, approximately 3.14159 [it takes an infinite number of decimal digits to get it exactly; this is because π is an *irrational number*[3] — *i.e.* one which cannot be expressed as a ratio of integers], defined in turn to be the ratio of the *circumference* ℓ of a circle to its *diameter* d: $\pi = \ell/d$ or $\ell = \pi d$.

Were you able to visualize all these simple plane (2-dimensional) shapes "in your head" without resort to actual drawings? If so, you may have a "knack" for geometry, if not Geometry. If it was confusing without the pictures, they are provided in Fig. 4.1 with the appropriate labels.

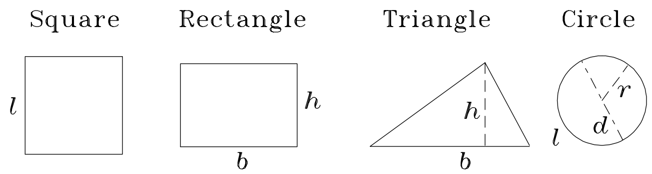

Figure 4.1 A few plane geometrical shapes, with labels.

4.2.2 The Pythagorean Theorem:

> The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the two shorter sides.

I.e. for the Left Hemisphere we have

$$c^2 = a^2 + b^2 \tag{1}$$

where a, b and c are defined by the labelled picture of a right triangle, shown in Fig. 4.2, which cathects the Right Hemisphere and gets the two working together.

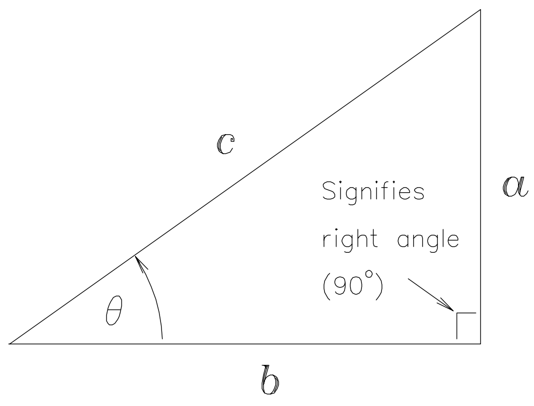

Figure 4.2 A right triangle with hypotenuse c and short sides a and b. The right angle is indicated and the angle θ is defined as shown. Note that a is always the (length of the) side "across from" the *vertex* forming the angle θ. This convention is essential in the *trigonometric* definitions to follow.

4.2.3 Solid Geometry

Most of us learned how to calculate the *volumes* of various solid or 3-dimensional objects even before we were told that the name for the system of conventions and "laws" governing such topics was "Solid Geometry." For instance, there is the *cube*, whose volume V is the *cube* (same chicken/egg problem again) of the length ℓ of one of its 8 *edges*: $V = \ell^3$. Similarly, a *cylinder* has a volume V equal to the product of its cross-sectional area A and its height h perpendicular to the base: $V = Ah$. Note that this works

[3] I do not know the proof that π is an irrational number, but I have been told by Mathematicians that it is, and I have never had any cause to question them. In principle, this is reprehensible (shame on me!) but I am not aware of any practical consequences one way or the other; if anyone knows one, please set me straight!

just as well for *any shape* of the cross-section — square, rectangle, triangle, circle or even some irregular oddball shape.

If you were fairly advanced in High School math, you probably learned a bit more abstract or general stuff about solids. But the really deep understanding that (I hope) you brought away with you was an awareness of the *qualitative* difference between 1-dimensional *lengths*, 2-dimensional *areas* and 3-dimensional *volumes*. This awareness can be amazingly powerful even without any "hairy Math details" if you consider what it implies about how these things change with *scale*.[4]

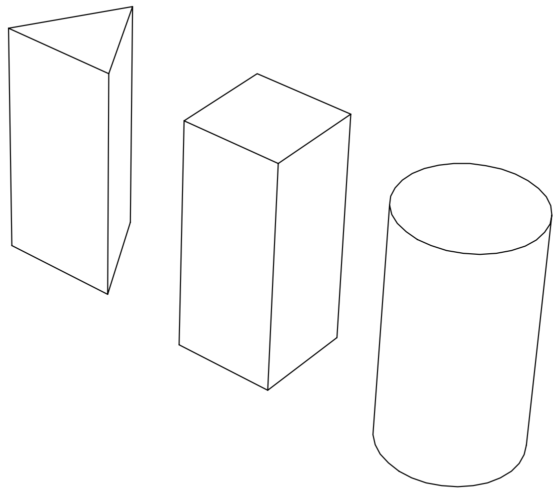

Figure 4.3 Triangular, square and circular right cylinders.

4.3 Algebra 1

A handy trick for introducing Algebra to young children (who have not yet learned that it is supposed to be too hard for them) is to phrase a typical Algebra problem in the following way: "I'm thinking of a number, and its name is 'x' ... so if $2x+3=7$, what is x?" (You may have to spend a little time explaining the notational conventions of equations and that $2x$ means 2 *times* x.) Most 7-year-olds can then solve this problem by inspection (my son and daughter both could!) but they may not be able to tell you *how* they solved it. This suggests either that early Arithmetic has already sown the seeds of algebraic manipulation conventions or that there is some understanding of such concepts "wired in" to our brains. We will never know how much of each is true, but certainly neither is entirely false!

[4]For instance, it explains easily why the largest animals on Earth have to live in the sea, why insects can lift so many times their own weight, why birds have an easier time flying than airliners, why bubbles form in beer and how the American nuclear power industry got off to a bad start. All in due time. . . .

What we learn in High School Algebra is to examine *how* we solve problems like this and to refine these techniques by adapting ourselves to a particular formalism and technology. Unfortunately our intuitive understanding is often trampled upon in the process — this happens when we are actively discouraged from treating the technology as a convenient representation for what we already understand, rather than a definition of correct procedure.

In Algebra we learn to "solve" equations. What does that mean? Usually it means that we are to take a (relatively) complicated equation that has the "unknown" (often but not always called "x") scattered all over the place and turn it into a (relatively) simple equation with x on the left-hand side by itself and a bunch of other symbols (*not* including x) on the right-hand side of the "=" sign. Obviously this particular *format* is "just" a convention. But the *idea* is independent of the representation: "solve" for the "unknown" quantity, in this case x.

There are a few basic rules we use to "solve" problems in Algebra; these are called "laws" by Mathematicians who want to emphasize that you are not to question their content or representation.

- **Definition of Zero:**

$$a - a = 0 \tag{2}$$

- **Definition of Unity:**

$$\frac{a}{a} = 1 \tag{3}$$

- **Commutative Laws:**[5]

$$a + b = b + a \tag{4}$$
and
$$ab = ba \tag{5}$$

- **Distributive Law:**

$$a(b + c) = ab + bc \tag{6}$$

- **Sum or Difference of Two Equations:** Adding (or subtracting) the same thing from both sides of an equation gives a new equation that is still OK.

$$\begin{array}{rrrcl} & x & -a = & b & \\ +\,(& & a = & a &) \\ & x & = & b + a & \end{array} \tag{7}$$

$$\begin{array}{rrrcl} & x & +c = & d & \\ -\,(& & c = & c &) \\ & x & = & d - c & \end{array} \tag{8}$$

- **Product or Ratio of Two Equations:** Multiplying (or dividing) both sides of an equation by the same thing also gives a new equation that is still OK.

$$\begin{array}{rrcl} & x/a = & b & \\ \times\,(& a = & a &) \\ & x = & ab & \end{array} \tag{9}$$

$$\begin{array}{rrcl} & cx = & d & \\ \div\,(& c = & c &) \\ & x = & d/c & \end{array} \tag{10}$$

These "laws" may seem pretty trivial (especially the first two) but they define the rules of Algebra whereby we learn to manipulate the form of equations and "solve" Algebra "problems." We quickly learn equivalent *shortcuts* like "moving a factor from the bottom of the left-hand-side [often abbreviated LHS] to the top of the right-hand side [RHS]:"

$$\frac{x - a}{b} = c + d \quad \Rightarrow \quad x - a = b(c + d) \tag{11}$$

and so on; but each of these is just a well-justified concatenation of several of the fundamental steps. *(Emergence!)*

You may ask, "Why go to so much trouble to express the obvious in such formal terms?" Well, as usual the obvious is not necessarily the truth. While the real, imaginary and complex numbers may all obey these simple rules, there are perfectly legitimate and useful fields of "things" (usually some sort of *operators*) that do *not* obey all these rules, as we shall see much later in the course (probably). It is generally a good idea to know your own assumptions; we haven't the time to keep reexamining them constantly, so we try to state them as plainly as we can and keep them around for reference "just in case...."

4.4 Trigonometry

Trigonometry is a specialized branch of Geometry in which we pay excruciatingly close attention to the properties of *triangles*, in particular *right* triangles. Referring to Fig. 4.2 again, we define the *sine* of the angle θ (abbreviated $\sin\theta$) to be the ratio of the "far side" a to the hypotenuse c and the *cosine* of θ (abbreviated $\cos\theta$) to be the ratio of the "near side" b to the

[5] Note that *division* is *not* commutative: $a/b \neq b/a$! Neither is *subtraction*, for that matter: $a - b \neq b - a$. The Commutative Law for *multiplication*, $ab = ba$, holds for ordinary numbers (real and imaginary) but it does *not* necessarily hold for all the mathematical "things" for which some form of "multiplication" is defined! For instance, the *group* of *rotation operators* in 3-dimensional space is *not* commutative — think about making two successive rotations of a rigid object about perpendicular axes in different order and you will see that the final result is different! This seemingly obscure property turns out to have fundamental significance. We'll talk about such things later.

hypoteneuse c:

$$\sin\theta \equiv \frac{a}{c} \qquad\qquad \cos\theta \equiv \frac{b}{c} \tag{12}$$

The other trigonometric functions can easily be defined in terms of the sin and cos:

tangent: $\tan\theta \equiv \frac{a}{b} = \frac{\sin\theta}{\cos\theta}$

cotangent: $\cot\theta \equiv \frac{b}{a} = \frac{\sin\theta}{\cos\theta} = \frac{1}{\tan\theta}$

secant: $\sec\theta \equiv \frac{c}{b} = \frac{1}{\cos\theta}$

cosecant: $\csc\theta \equiv \frac{c}{a} = \frac{1}{\sin\theta}$

For the life of me, I can't imagine why anyone invented the *cotangent*, the *secant* and the *cosecant* — as far as I can tell, they are totally superfluous baggage that just slows you down in any actual calculations. Forget them. [Ahhhh. I have always wanted to say that! Of course you are wise enough to take my advice with a grain of salt, especially is you want to appear clever to Mathematicians. . . .]

The *sine* and *cosine* of θ are our trigonometric workhorses. In no time at all, I will be wanting to think of them as *functions* — *i.e.* when you see "$\cos\theta$" I will want you to say, "cosine of theta" and think of it as $\cos(\theta)$ the same way you think of $y(x)$. Whether as simple ratios or as functions, they have several delightful properties, the most important of which is obvious from the Pythagorean Theorem:[6]

$$\cos^2\theta + \sin^2\theta = 1 \tag{13}$$

where the notation $\sin^2\theta$ means the *square* of $\sin\theta$ — *i.e.* $\sin^2\theta \equiv (\sin\theta)\times(\sin\theta)$ — and similarly for $\cos\theta$. This convention is adopted to avoid confusion, believe it or not. If we wrote "$\sin\theta\ ^2$" it would be impossible to know for sure whether we meant $\sin(\theta^2)$ or $(\sin\theta)^2$; we could always put parentheses in the right places to remove the ambiguity, but in this case there is a convention instead. (People always have conventions when they are tired of thinking!)

I will need other trigonometric identities later on, but they can wait — why introduce math until we need it? [I have made an obvious exception in this Chapter as a whole only to "jump start" your Mathematical language (re)training.]

4.5 Algebra 2

"I'm thinking of a number, and its name is 'x' . . . " So if

$$ax^2 + bx + c = 0, \tag{14}$$

what is x? Well, we can only say, "It depends." Namely, it depends on the values of a, b and c, whatever they are. Let's suppose the *dimensions* of all these "parameters" are mutually consistent[7] so that the equation makes sense. Then "it can be shown" (a classic phrase if there ever was one!) that the "answer" is *generally*[8]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{15}$$

This formula (and the preceding equation that defines what we mean by a, b and c) is known as the *Quadratic Theorem*, so called because it offers "the answer" to *any* quadratic equation (*i.e.* one containing powers of x up to and including x^2). The power of such a *general* solution is prodigious. (Work out a few examples!) It also introduces an interesting new way of looking at

[6] Surely you aren't going to take my word for this! *Convince yourself* that this formula is really true!

[7] In Mathematics we never worry about such things; all our symbols represent *pure numbers*; but in Physics we *usually* have to express the value of some physical quantity in units which make sense and are consistent with the units of other physical quantities symbolized in the same equation!

[8] The $\pm$ symbol means that *both* signs (+ and −) should represent legitimate answers.

the relationship between x and the *parameters* a, b and c that determine its value(s). Having x all by itself on one side of the equation and no x's anywhere on the other side is what we call a "solution" in Algebra. Let's make a simpler version of this sort of equation:

"I'm thinking of a number, and its name is 'y' ..." So if $y = x^2$, what is y? The answer is again, "It depends!" (In this case, upon the value of x.) And that leads us into a new subject....

4.6 Calculus

In a *stylistic* sense, Algebra starts to become Calculus when we write the preceding example, $y = x^2$, in the form

$$y(x) = x^2$$

which we read as "y of x equals x squared." This is how we signal that we mean to think of y as a *function* of x, and right away we are leading into the terminology of Calculus. Recall the final sections of the preceding Chapter.

However, Calculus really begins when we start talking about the *rate of change* of y as x varies.

4.6.1 Rates of Change

One thing that is easy to "read off a graph" of $y(x)$ is the *slope* of the curve at any given point x. Now, if $y(x)$ is quite "curved" at the point of interest, it may seem contradictory to speak of its "slope," a property of a *straight* line. However, it is easy to see that as long as the curve is *smooth* it will always *look like a straight line* under sufficiently high *magnification*. This is illustrated in Fig. 4.4 for a typical $y(x)$ by a process of successive magnifications.

We can also prescribe an algebraic method for *calculating* the slope, as illustrated in Fig. 4.5:

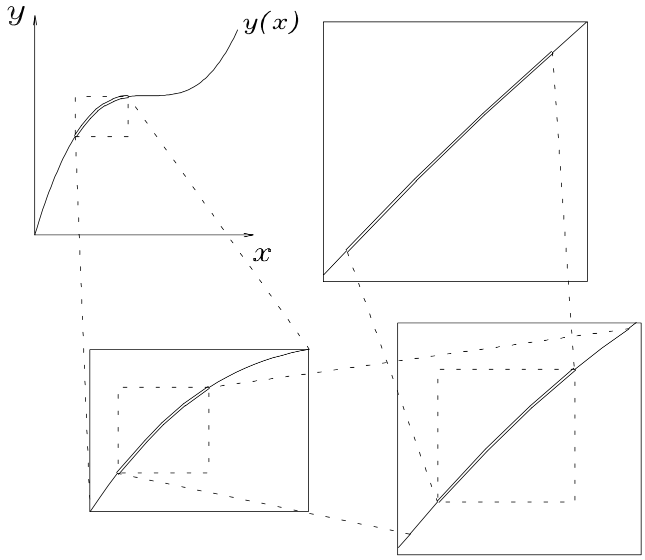

Figure 4.4 A series of "zooms" on a segment of the curve $y(x)$ showing how the *curved* line begins to look more and more like a *straight* line under higher and higher magnification.

the *definition* of the "slope" is the ratio of the increase in y to the increase in x on a vanishingly small interval. That is, when x goes from its initial value x_0 to a slightly larger value $x_0 + \Delta x$, the curve carries y from its initial value $y_0 = y(x_0)$ to a new value $y_0 + \Delta y = y(x_0 + \Delta x)$, and the slope of the curve at $x = x_0$ is given by $\Delta y / \Delta x$ for a vanishingly small Δx. When a small change like Δx gets *really* small (*i.e.* small enough that the curve looks like a straight line on that interval, or "small enough to satisfy whatever criterion you want," then we write it differently, as dx, a "*differential*" (vanishingly small) change in x. Then the exact definition of the SLOPE of y with respect to x at some particular value of x, written in conventional Mathematical language, is

$$\frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \quad (16)$$

This is best understood by an example: consider the simple function $y(x) = x^2$. Then

$$y(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$

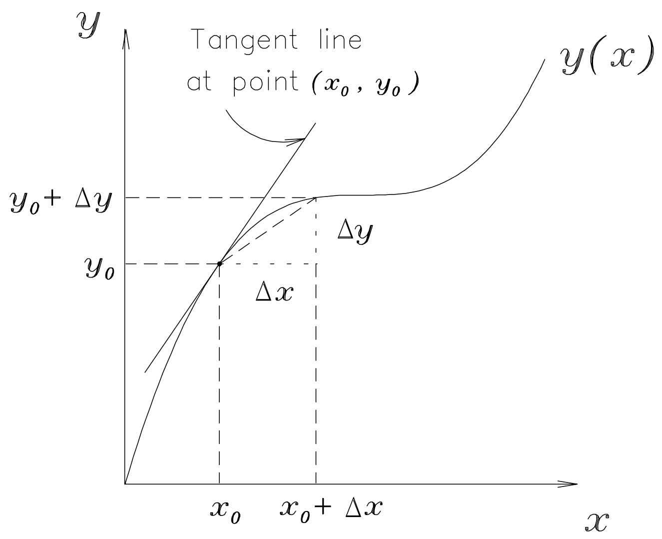

Figure 4.5 A graph of the function $y(x)$ showing how the *average* slope $\Delta y/\Delta x$ is obtained on a *finite* interval of the curve. By taking smaller and smaller intervals, one can eventually obtain the slope at a *point*, dy/dx.

$$\text{and} \quad y(x + \Delta x) - y(x) = 2x\Delta x + (\Delta x)^2.$$

Divide this by Δx and we have

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

Now let Δx shrink to zero, and all that remains is

$$\frac{\Delta y}{\Delta x} \underset{\Delta x \to 0}{\longrightarrow} \frac{dy}{dx} = 2x.$$

Thus the slope [or *derivative*, as mathematicians are wont to call it] of $y(x) = x^2$ is $dy/dx = 2x$. That is, the slope increases linearly with x. The slope *of* the slope — which we call the *curvature*, for obvious reasons — is then trivially $d(dy/dx)/dx \equiv d^2y/dx^2 = 2$, a constant. Make sure you can work this part out for yourself.

We have defined all these algebraic solutions to the geometrical problem of finding the slope of a curve on a graph in completely abstract terms — "x" and "y" indeed! What *are* x and y? Well, the whole idea is that they can be anything you want! The most common examples in Physics are when x is the *elapsed time*, usually written t, and y is the *distance travelled*, usually (alas) written x. Thus in an elementary Physics context the function you are apt to see used most often is $x(t)$, the position of some object as a function of time. This particular function has some very well-known derivatives, namely $dx/dt = v$, the *speed* or (as long as the motion is in a straight line!) *velocity* of the object; and $dv/dt \equiv d^2x/dt^2 = a$, the *acceleration* of the object. Note that both v and a are themselves (in general) functions of time: $v(t)$ and $a(t)$. This example so beautifully illustrates the "meaning" of the slope and curvature of a curve as first and second derivatives that many introductory Calculus courses and virtually all introductory Physics courses use it as *the* example to explain these Mathematical conventions. I just had to be different and start with something a little more formal, because I think you will find that the idea of one thing being a *function* of another thing, and the associated ideas of graphs and slopes and curvatures, are handy notions worth putting to work far from their traditional realm of classical kinematics.